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Fibre bundle formulation of nonrelativistic quantum mechanics: I. Introduction. The evolution transport

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Abstract

We propose a new systematic fibre bundle formulation of nonrelativistic quantum mechanics. The new form of the theory is equivalent to the usual one and is in harmony with the modern trends in theoretical physics and potentially admits new generalizations in different directions. In it the Hilbert space of a quantum system (from conventional quantum mechanics) is replaced with an appropriate Hilbert bundle of states and a pure state of the system is described by a lifting of paths or sections along paths in this bundle. The evolution of a pure state is determined through the bundle (analogue of the) Schrödinger equation. Now the dynamical variables and density operators are described via liftings of paths or morphisms along paths in suitable bundles. The mentioned quantities are connected by a number of relations derived in this paper.

The present, first, part of this investigation is devoted to the introduction of basic concepts on which the fibre bundle approach to quantum mechanics rests. We show that the evolution of pure quantum mechanical states can be described as a suitable linear transport along paths, called evolution transport, of the state liftings in the Hilbert bundle of states of a considered quantum system.

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1. Introduction

Usually the standard nonrelativistic quantum mechanics of pure states is formulated in terms of vectors and operators in a Hilbert space [1–5]. This is in discord and not in harmony with the new trends in (mathematical) physics [6–8] in which the theory of fibre bundles [9, 10], in particular vector bundles [11, 12], is essentially used. This paper (and

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its further continuation(s)) is intended to incorporate the quantum theory in the family of fundamental physical theories based on the background of fibre bundles.

The idea of geometrization of quantum mechanics is an old one (see, for example, [13] and references therein). A good motivation for such an approach is given in [13, 14]. Different geometrical structures in quantum mechanics were introduced [15, 16], for instance inner products(s) [2, 3, 13, 17], (linear) connection [14, 17, 18], symplectic structure [14], complex structure [13] etc. The introduction of such structures admits a geometrical treatment of some problems, for instance, the dynamics in the (quantum) phase space [13] and the geometrical phase [14]. In a very special case, a gauge structure, i.e. a parallel transport corresponding to a linear connection, is pointed out in quantum mechanics in [19]. For us this paper is remarkable for the fact that equation (10) in it is a very 'ancient' special version of the transformation law for the matrix-bundle Hamiltonian, derived in this investigation. It, together with the bundle (analogue of the) Schrödinger equation, shows that (up to a constant) the Hamiltonian plays the role of a gauge field (linear connection) with respect to the quantum evolution. In [18, 20] one finds different (vector) bundles defined on the base of the (usual) Hilbert space of quantum mechanics or its modifications. In these works different parallel transports in the corresponding bundles are introduced too. Some interesting ideas concerning the theory of fibre bundles and quantum mechanics can be found in [21].

In the 1970s a whole theory, called geometric quantization, was developed to clarify the relations between classical and quantum mechanics in geometrical terms. It started from the early works [22–26] although some of its basic ideas were presented in [27, 28]. A recent review of the foundations of geometric quantization can be found in [29] (see also the references in this paper). Since the geometric quantization concerns only the kinematical aspects of the quantum theory (on a basis quite different from ours), it will not be treated in our exposition, but it is worth mentioning that this theory contains important geometrical structures such as the symplectic one and vector (in particular line) bundles and linear connections in them (see, e.g., [30, 31]).

A general feature of all of the references cited above is that in them all geometric concepts are introduced by using in one way or another the accepted mathematical foundation of quantum mechanics, namely a suitable Hilbert or projective Hilbert space and operators acting in it. The Hilbert space may be extended in a certain sense or replaced by a more general space, but this does not change the main ideas. One of the aims of this paper is thus to change this mathematical background of quantum mechanics.

Separately we have to mention the approach of Prugovečki to the quantum theory, a selective summary of which can be found in [32] (see also references therein) and in [33]. It can be characterized as 'stochastic' and 'bundle'. The former feature will not be discussed in the present investigation; thus we lose some advantages of the stochastic quantum theory to which we shall return elsewhere. The latter 'part' of Prugovečki's approach has some common aspects with our present work but, generally, it is essentially different. For instance, in both cases the quantum evolution from point to point (in spacetime) is described via a kind of (parallel or generic linear) transport (along paths) in a suitable Hilbert fibre bundle, but the notion of a 'Hilbert bundle' in our approach and that of Prugovečki is different regardless of the fact that in both cases the typical (standard) fibre is practically the same (when one and the same theory is concerned). Besides, we do not need even to introduce the Poincaré (principal) fibre bundle over the spacetime or phase space which plays an important role in Prugovečki's theory. Also we have to notice that the concepts of quantum and parallel transport used in it are special cases of the notion of a 'linear transport along paths' introduced in [34, 35]. The application of the last concept, which is accepted in the present investigation, has many advantages, significantly simplifies some proofs and makes certain results 'evident' or trivial (e.g. the last part of section 2 and the whole section 4 of [32]). Finally, at the present level (nonrelativistic quantum mechanics), our bundle formulation of the quantum theory is insensitive with respect to the spacetime curvature. A detailed comparison of our approach to the quantum theory and that of Prugovečki will be made elsewhere.

Another geometric approach to quantum mechanics is proposed in [36] and partially in [37]; the latter is, with a few exceptions, almost a review of the former. These works suggest two ideas which are quite important for us. First, the quantum evolution could be described as a (kind of) parallel transport in an infinite-dimensional (Hilbert) fibre bundle over the spacetime. Second, the concrete description of a quantum system should explicitly depend on (the state of) the observer with respect to which it is depicted (or who 'investigates' it). These ideas are incorporated and developed in our paper.

From the literature known to the author, the work [38] is closest to the approach developed here. It contains an excellent motivation for applying the fibre bundle technique to nonrelativistic quantum mechanics[2]. Generally, in this paper the evolution of a quantum system is described as a 'generalized parallel transport' of appropriate objects in a Hilbert fibre bundle over the one-dimensional manifold $\mathbb{R}_+ := \{t : t \in \mathbb{R}, t \geqslant 0\}$, interpreted as a 'time' manifold (space). We shall comment on [38] in the second part of this series, after developing the formalism required for its analysis. Besides, we emphasize once again, the paper [38] contains an excellent description of why the apparatus of fibre bundle theory is a natural scene for quantum mechanics.

An attempt to formulate quantum mechanics in terms of a fibre bundle over the phase space is made in [39]. Regardless of some common features, this paper is quite different from the present investigation. We shall comment on it later. In particular, in [39] the gauge structure of the arising theory is governed by a nondynamical connection related to the symplectic structure of the system's phase space, while in this paper analogous structure (linear transport along paths) is uniquely connected with the system's Hamiltonian, playing here the rôle of a gauge field itself.

We should also mention a recent approach to quantum mechanics, called covariant quantum mechanics, developed at the beginning of the 1990s by Modugno and Jadczyk [40–44] (see references therein too), based on jets, connections and cosymplectic forms. It utilizes the basic 'bundle' idea of [38] and employs two geometrical structures that are similar to those we intend to use in this paper: a quantum bundle, which is a complex line bundle (i.e. one-dimensional vector bundle) over spacetime (equipped with Hermitian metric), and a Hilbert bundle over a time manifold (realized as $\mathbb{R}_+$ or $\mathbb{R}$). Generally, the covariant quantum mechanics is quite different from the theory we are going to develop in the present exposition. However, the cited references, especially [43], contain a good motivation why the fibre bundle formalism is a natural one for describing quantum mechanical events. We should also emphasize the treatment of quantum mechanical evolution as a parallel transport in a Hilbert bundle over time (manifold) in [43] which is quite similar to our understanding of this phenomenon in the present paper.

This paper is a direct continuation of the considerations in [45], which paper, in fact, may be regarded as its preliminary version. Here we suggest a *purely fibre bundle formulation of the nonrelativistic quantum mechanics*. The proposed geometric formulation of quantum mechanics is *dynamical* in a sense that all geometrical structures employed for the description of a quantum system depend on and are determined by the dynamical characteristics of the system. This new form of the theory is *entirely equivalent* to the usual one, which is a consequence or our step by step equivalent reformulation of the quantum theory. The bundle description is obtained on the base of the developed by the author theory of transports along

[2] The author thanks Dr J F Cariñena (University of Zaragoza) for drawing his attention to [38] in May 1998.

paths in fibre bundles [34, 35, 46], generalizing the theory of parallel transport. It is partially generalized here to the infinite-dimensional case.

The main object in quantum mechanics is the Hamiltonian (operator) which, through the Schrödinger equation, governs the evolution of a quantum system [2–5]. In our novel approach its role is played by a suitable linear transport along paths in an appropriate Hilbert bundle. It turns out that up to a constant the matrix-bundle Hamiltonian, which is uniquely determined by the Hamiltonian in a given field of bases, coincides with the matrix of the coefficients of this transport (cf an analogous result in [45, section 5]). This fact, together with the replacement of the usual Hilbert space with a Hilbert bundle, is the corner-stone for the possibility for the new formulation of the nonrelativistic quantum mechanics.

The present, first, part of our investigation is organized as follows.

In section 2 are reviewed some well known facts from the standard quantum mechanics and our notation is partially fixed. Here, as well as throughout this paper, we follow the degree of rigour established in the physical literature, but, if required, this paper can be reformulate to meet the present-day mathematical standards. For this purpose one can use, for instance, the quantum mechanical formalism described in [4] or in [47] (see also [48, 49]).

Section 3 contains preliminary mathematical material required for the goals of this paper. In section 3.1 are collected some basic definitions concerning vector and Hilbert bundles. Next, in section 3.2, the notion of a linear transport along paths in vector bundles is recalled and some its peculiarities in the Hilbert bundle case are pointed out. In section 3.3 the concepts of liftings of paths, sections along paths and derivations along paths are introduced.

Section 4 begins the building of the new bundle approach to quantum mechanics. After a brief review of some references (section 4.1), a motivation for the application of fibre bundle formalism to quantum mechanics is presented (section 4.2). Also some heuristic considerations of elements of the new theory are given. Section 4.3 introduces the basic initial assertions of the bundle version of quantum mechanics. They are formulated in a form of two postulates which are enough for the bundle description of the evolution of a quantum system. In the bundle approach the analogues of the Hilbert space of states and state vector of a system are the systems Hilbert bundle (of states) and the (state) lifting of paths (or sections along paths) in it. Preliminary summary of some results of this investigation is presented in section 4.4.

In section 5 it is proved that the evolution operator of a quantum system is (equivalently) replaced in the bundle description by a suitable linear transport along paths, called *evolution transport*.

The paper closes with section 6.

2. Evolution of pure quantum states (review)

In quantum mechanics [2, 3, 5, 49] a pure state of a quantum system is described by state vector $\psi(t)$ (in Dirac's [1] notation $|t\rangle$). Generally, it depends on the time $t \in \mathbb{R}$ and belongs to a Hilbert space $\mathcal{F}$ (specific to any concrete system) generically endowed with a nondegenerate Hermitian scalar product[3] $\langle\cdot|\cdot\rangle : \mathcal{F} \times \mathcal{F} \to \mathbb{C}$. For every two instants of time t_2 and t_1 the corresponding state vectors are connected by the equality

$$\psi(t_2) = \mathcal{U}(t_2, t_1)\psi(t_1) \tag{2.1}$$

[3] For some (e.g. unbounded) states the system's state vectors form a more general space than a Hilbert one. This is insignificant for the following presentation. A sufficient summary, for our purposes, of Hilbert space theory is given in [50, appendix].

where $\mathcal{U}$ is the *evolution operator* of the system [4, chapter 4, section 3.2]. It is supposed to be linear and unitary, i.e.

$$\mathcal{U}(t_2, t_1)(\lambda\psi(t_1) + \mu\xi(t_1)) = \lambda\mathcal{U}(t_2, t_1)(\psi(t_1)) + \mu\mathcal{U}(t_2, t_1)(\xi(t_1)) \tag{2.2}$$

$$\mathcal{U}^\dagger(t_1, t_2) = \mathcal{U}^{-1}(t_2, t_1) \tag{2.3}$$

for every $\lambda, \mu \in \mathbb{C}$ and state vectors $\psi(t), \xi(t) \in \mathcal{F}$, and such that for any t

$$\mathcal{U}(t, t) = \mathsf{id}_\mathcal{F}. \tag{2.4}$$

Here id_X means the identity map of a set X and the dagger (†) denotes Hermitian conjugation, i.e. if $\varphi, \psi \in \mathcal{F}$ and $\mathcal{A} : \mathcal{F} \to \mathcal{F}$, then $\mathcal{A}^\dagger$ is defined by

$$\langle \mathcal{A}^\dagger\varphi|\psi\rangle = \langle\varphi|\mathcal{A}\psi\rangle. \tag{2.5}$$

In particular $\mathcal{U}^\dagger$ is defined by $\langle\mathcal{U}^\dagger(t_1, t_2)\varphi(t_2)|\psi(t_1)\rangle = \langle\varphi(t_2)|\mathcal{U}(t_2, t_1)\psi(t_1)\rangle$. So, knowing $\psi(t_0) = \psi_0$ for some moment t_0, one knows the state vector for every moment t as $\psi(t) = \mathcal{U}(t, t_0)\psi(t_0) = \mathcal{U}(t, t_0)\psi_0$.

Let $\mathcal{H}(t)$ be the Hamiltonian (function) of a system. It generally depends on the time t explicitly[4] and it is supposed to be a Hermitian operator, i.e. $\mathcal{H}^\dagger(t) = \mathcal{H}(t)$. The Schrödinger equation (see [1, section 27] or [4, chapter 5, section 3.1])

$$\mathrm{i}\hbar\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = \mathcal{H}(t)\psi(t) \tag{2.6}$$

with $\mathrm{i} \in \mathbb{C}$ and $\hbar$ being, respectively, the imaginary unit and Plank's constant (divided by 2π), together with some initial condition

$$\psi(t_0) = \psi_0 \in \mathcal{F} \tag{2.7}$$

is postulated in the quantum mechanics. They determine the time evolution of a state vector $\psi(t)$.

The substitution of (2.1) into (2.6) shows that there is a bijective correspondence between $\mathcal{U}$ and $\mathcal{H}$ described by

$$\mathrm{i}\hbar\frac{\partial\mathcal{U}(t, t_0)}{\partial t} = \mathcal{H}(t) \circ \mathcal{U}(t, t_0) \qquad \mathcal{U}(t_0, t_0) = \mathsf{id}_\mathcal{F} \tag{2.8}$$

where $\circ$ denotes composition of maps. If $\mathcal{U}$ is given, then

$$\mathcal{H}(t) = \mathrm{i}\hbar\frac{\partial\mathcal{U}(t, t_0)}{\partial t} \circ \mathcal{U}^{-1}(t, t_0) = \mathrm{i}\hbar\frac{\partial\mathcal{U}(t, t_0)}{\partial t} \circ \mathcal{U}(t_0, t) \tag{2.9}$$

where we have used the equality

$$\mathcal{U}^{-1}(t_2, t_1) = \mathcal{U}(t_1, t_2)$$

which follows from (2.1) (see also below (2.10) or section 5). Conversely, if $\mathcal{H}$ is given, then [3, chapter 8, section 8] $\mathcal{U}$ is the unique solution of the integral equation $\mathcal{U}(t, t_0) = \mathsf{id}_\mathcal{F} + \frac{1}{\mathrm{i}\hbar}\int_{t_0}^{t}\mathcal{H}(\tau)\mathcal{U}(\tau, t_0)\,\mathrm{d}\tau$, i.e. we have

$$\mathcal{U}(t, t_0) = \mathrm{Texp}\int_{t_0}^{t}\frac{1}{\mathrm{i}\hbar}\mathcal{H}(\tau)\,\mathrm{d}\tau \tag{2.10}$$

where $\mathrm{Texp}\int_{t_0}^{t}\cdots\,\mathrm{d}\tau$ is the chronological (called also T-ordered, P-ordered or path-ordered) exponent (defined, for example, as the unique solution of the initial-value problem (2.8); see

[4] Of course, the Hamiltonian also depends on the observer with respect to which the evolution of the quantum system is described. This dependence is usually implicitly assumed and not written explicitly [1, 3]. This deficiency will be eliminated in a natural way later in this paper. The Hamiltonian can also depend on other quantities, such as the (operators of the) system's generalized coordinates. This possible dependence is insignificant for our investigation and will not be written explicitly.

also [38, equation (1.3)])[5]. From here it follows that the Hermiticity of $\mathcal{H}$, $\mathcal{H}^\dagger = \mathcal{H}$, is equivalent to the unitarity of $\mathcal{U}$ (see (2.3)).

Let us note that for the rigorous mathematical understanding of the derivations in (2.6), (2.8), and (2.9), as well as of the chronological (path-ordered) exponent in (2.10), one has to apply the mathematical apparatus developed in [4], but this is outside the scope of this paper.

If $\mathcal{A}(t) : \mathcal{F} \to \mathcal{F}$ is the (linear Hermitian) operator corresponding to a dynamical variable $\mathbb{A}$ at the moment t, then the mean value (= the mathematical expectation) which it assumes at a state described by a state vector $\psi(t)$ with a finite norm is

$$\langle \mathcal{A}\rangle_\psi^t := \langle \mathcal{A}(t)\rangle_\psi^t := \langle \mathcal{A}(t)\rangle_{\psi(t)} := \frac{\langle \psi(t)|\,\mathcal{A}(t)\psi(t)\rangle}{\langle \psi(t)|\psi(t)\rangle}. \tag{2.11}$$

This is interpreted as the observed value of $\mathbb{A}$ that can be measured experimentally.

Often the operator $\mathcal{A}$ can be chosen independent of the time t. (This is possible, for example, if $\mathcal{A}$ does not depend on t explicitly [3, chapter 7, section 9] or if the spectrum of $\mathcal{A}$ does not change in time [2, chapter 3, section 13].) If this is the case, it is said that the system's evolution is depicted in the Schrödinger picture of motion [1, section 28], [3, chapter 7, section 9].

3. Mathematical preliminaries

Before starting with the formulation of quantum mechanics in terms of fibre bundles, several geometrical tools have to be known. In this section is collected most of the pure mathematical material required for this goal. First, we present some facts from the theory of Hilbert bundles. These bundles will replace the Hilbert spaces in quantum mechanics. Next, we give a brief introduction to the theory of linear transports along paths in vector bundles and specify peculiarities of the Hilbert bundle case. The linear transports are needed for the bundle description of quantum evolution. Finally, we pay attention to liftings of paths and section along paths. These objects will replace the state vectors of ordinary quantum mechanics.

3.1. Hilbert bundles

At the beginning, to fix the terminology, we recall the definitions of bundle, section, fibre map, morphism and vector bundle[6].

A *bundle* is a triple (E, π, B) of sets E and B, called the (total) bundle space and the base (space) respectively, and (generally) surjective mapping $\pi : E \to B$, called the projection. For every $b \in B$ the set $\pi^{-1}(b)$ is called the fibre over b. If $X \subseteq B$, the bundle $(E, \pi, B)|_X := (\pi^{-1}(X), \pi|_{\pi^{-1}(X)}, X)$ is called the restriction on X of a bundle (E, π, B). A *section* of the bundle (E, π, B) is a mapping $\sigma : B \to E$ such that $\pi \circ \sigma = \mathsf{id}_B$; i.e., $\sigma : b \mapsto \sigma(b) \in \pi^{-1}(b)$. The set of sections of (E, π, B) is denoted by $\mathrm{Sec}(E, \pi, B)$.

A mapping $\varphi : E \to E$ is said to be a *fibre map* if it carries fibres into fibres. Precisely, φ is a fibre map iff, for every $b \in B$, there exists a point $b' \in B$ such that φ maps $\pi^{-1}(b)$ into $\pi^{-1}(b')$; i.e., $\varphi|_{\pi^{-1}(b)} : \pi^{-1}(b) \to \pi^{-1}(b')$. A *(fibre) morphism* of the bundle (E, π, B) is a pair (φ, f) of maps $\varphi : E \to E$ and $f : B \to B$ such that $\pi \circ \varphi = f \circ \pi$. The set of morphisms of (E, π, B) is denoted by $\mathrm{Mor}(E, \pi, B)$, i.e.

$$\mathrm{Mor}(E, \pi, B) := \{(\varphi, f)|\varphi : E \to E,\ f : B \to B,\ \pi \circ \varphi = f \circ \pi\}.$$

[5] The physical meaning of $\mathcal{U}$ as a propagation function, as well as its explicit calculation (in component form) via $\mathcal{H}$ can be found, for example, in [51, sections 21 and 22].

[6] For details, examples etc, see [9, 11, 12, 52–55].

A map $\varphi : E \to E$ is called a *morphism over B* or *B-morphism* if $(\varphi, \mathrm{id}_B) \in \mathrm{Mor}(E, \pi, B)$. The set of all B-morphisms of (E, π, B) will be denoted by $\mathrm{Mor}_B(E, \pi, B)$; i.e.,

$$\mathrm{Mor}_B(E, \pi, B) := \{\varphi | \varphi : E \to E,\ \pi \circ \varphi = \pi\}.$$

For every morphism $(\varphi, f) \in \mathrm{Mor}(E, \pi, B)$, the map φ is a fibre map since from $\pi \circ \varphi = f \circ \pi$ it follows that $\varphi|_{\pi^{-1}(b)} : \pi^{-1}(b) \to \pi^{-1}(f(b))$ for every $b \in B$. In particular, any B-morphism $\varphi \in \mathrm{Mor}_B(E, \pi, B)$ is a fibre-preserving map as $\varphi|_{\pi^{-1}(b)} : \pi^{-1}(b) \to \pi^{-1}(b)$. Conversely, every fibre map $\varphi : E \to E$ defines a morphism (φ, f) with $f := \pi \circ \varphi \circ \pi^{-1} : B \to B$; f is called the *induced map* of the fibre map φ, and (φ, f) is the *induced morphism*[7].

Consider the set of *point-restricted morphisms*

$$\begin{aligned} E_0 : &= \{(\varphi_b, f) \mid \varphi_b = \varphi|_{\pi^{-1}(b)},\ b \in B,\ (\varphi, f) \in \mathrm{Mor}(E, \pi, B)\} \\ &= \{(\varphi_b, f) \mid \varphi_b : \pi^{-1}(b) \to \pi^{-1}(f(b)),\ b \in B,\ f : B \to B\} \end{aligned}$$

i.e. $(\psi, f) \in E_0$ iff $f : B \to B$ and there exists a unique $b \in B$ such that $\psi : \pi^{-1}(b) \to \pi^{-1}(f(b))$ and we write ψ_b for $\psi|_{\pi^{-1}(b)}$. Defining $\pi_0 : E_0 \to B$ by $\pi_0(\varphi_b, f) := b$ for $(\varphi_b, f) \in E_0$, we see that (E_0, π_0, B) is a bundle over the same base B as (E, π, B). This is the *bundle of point-restricted morphisms* of (E, π, B). It will be denoted by $\mathrm{mor}(E, \pi, B)$; i.e., $\mathrm{mor}(E, \pi, B) := (E_0, \pi_0, B)$.

There exists a bijective correspondence τ such that

$$\mathrm{Mor}(E, \pi, B) \xrightarrow{\tau} \mathrm{Sec}\big(\mathrm{mor}(E, \pi, B)\big).$$

In fact, for $(\varphi, f) \in \mathrm{Mor}(E, \pi, B)$, we put $\tau : (\varphi, f) \mapsto \tau_{(\varphi, f)}$ with $\tau_{(\varphi, f)} : b \mapsto \tau_{(\varphi, f)}(b) := (\varphi|_{\pi^{-1}(b)}, f) \in \pi_0^{-1}(b)$ for every $b \in B$. Conversely, for $\sigma \in \mathrm{Sec}(\mathrm{mor}(E, \pi, B))$, we set $\tau^{-1} : \sigma \mapsto \tau^{-1}(\sigma) := (\varphi, f) \in \mathrm{Mor}(E, \pi, B)$, where, if $b \in B$ and $\sigma(b) = (\varphi_b, f)$, the map $\varphi : E \to E$ is defined by $\varphi|_{\pi^{-1}(b)} := \varphi_b$.

The above constructions can be modified for morphisms over the bundle base as follows. The *bundle* $\mathrm{mor}_B(E, \pi, B)$ *of point-restricted morphisms over B* of (E, π, B) has a base B, bundle space

$$\begin{aligned} E_0^B : &= \{\varphi_b \mid \varphi_b = \varphi|_{\pi^{-1}(b)},\ b \in B,\ \varphi \in \mathrm{Mor}_B(E, \pi, B)\} \\ &= \{\varphi_b \mid \varphi_b : \pi^{-1}(b) \to \pi^{-1}(b),\ b \in B\} \end{aligned}$$

and projection $\pi_0^B : E_0^B \to B$ such that

$$\pi_0^B(\varphi_b) := b \qquad \varphi_b \in E_0^B.$$

For brevity, the bundle $\mathrm{mor}_B(E, \pi, B)$ will be referred to as the *bundle of restricted morphisms* of (E, π, B). Evidently, the set E_0^B coincides with the set of point-restricted fibre-preserving fibre maps of (E, π, B). There is a bijection

$$\mathrm{Mor}_B(E, \pi, B) \xrightarrow{\chi} \mathrm{Sec}\big(\mathrm{mor}_B(E, \pi, B)\big)$$

given by $\chi : \varphi \mapsto \chi_\varphi$, $\varphi \in \mathrm{Mor}_B(E, \pi, B)$, with $\chi_\varphi : b \mapsto \chi_\varphi(b) := \varphi|_{\pi^{-1}(b)}$, $b \in B$. Its inverse is $\chi^{-1} : \sigma \mapsto \chi^{-1}(\sigma) = \varphi$, $\sigma \in \mathrm{Sec}(\mathrm{mor}_B(E, \pi, B))$, with $\varphi : E \to E$ given via $\varphi|_{\pi^{-1}(b)} = \sigma(b)$ for every $b \in B$.

If E and B are topological spaces, which is the most widely considered case, the bundle (E, π, B) is called *topological*. In this case in the definition of a bundle is included the *bundle property*: there exists a (topological) space $\mathcal{E}$ such that for each $b \in B$ there is an open neighbourhood ('directory space') W of b in B and homeomorphism ('decomposition function') $\phi_W : W \times \mathcal{E} \to \pi^{-1}(W)$ of $W \times \mathcal{E}$ onto $\pi^{-1}(W)$ satisfying the condition $(\pi \circ \phi_W)(w, e) = w$ for $\psi \in W$ and $e \in \mathcal{E}$; i.e., $\pi \circ \phi_W = \mathrm{id}_W$. Besides, if the restriction

[7] The transport along a path $\gamma : J \to B$ is an example of a fibre map along γ—*vide infra* section 3.2.

$\phi_W|_b : \{b\} \times \mathcal{E} \to \pi^{-1}(b)$, $b \in B$, is a homeomorphism, the bundle property is called *local triviality*, $\mathcal{E}$ is called a *(typical, standard) fibre of the bundle*, the bundle is called *locally trivial* and every fibre $\pi^{-1}(b)$ is homeomorphic to $\mathcal{E}$ for every $b \in B$.

A *vector bundle* is a locally trivial bundle (E, π, B) such that (i) the fibres $\pi^{-1}(b)$, $b \in B$, and the standard fibre $\mathcal{E}$ are (linearly) isomorphic vector spaces and (ii) the decomposition mappings ϕ_W and their restrictions $\phi_W|_b$ are (linear) isomorphisms between vector spaces. The dimension of $\mathcal{E}$, $\dim \mathcal{E} = \dim \pi^{-1}(b)$ for every $b \in B$, is called the dimension of the vector bundle; it is called $\dim \mathcal{E}$-dimensional. Here the vector spaces are considered over some field, usually the real or complex numbers; in the context of this paper, the complex case will be employed.

When vector bundles are considered, in the definition of a morphism or B-morphism is included the condition that the corresponding fibre maps are *linear*. For example, $\varphi : E \to E$ is morphism over B of a vector bundle (E, π, B) if $\pi \circ \varphi = \pi$ and the restricted mapping $\varphi|_{\pi^{-1}(b)} : \pi^{-1}(b) \to \pi^{-1}(b)$ is linear for every $b \in B$.

Definition 3.1. *A Hilbert (fibre) bundle is a vector bundle whose fibres over the base are isomorphic Hilbert spaces or, equivalently, whose (standard) fibre is a Hilbert space.*

In the present investigation we shall show that the Hilbert bundles can be taken as a natural mathematical framework for a geometrical formulation of quantum mechanics.

Some quite general aspects of the Hilbert bundles can be found in [50, chapter 7]. Below we are going to consider only certain specific properties and structures of the Hilbert bundle theory required for the present investigation.

Let (F, π, M) be a Hilbert bundle with bundle space F, base M, projection π and (typical) fibre $\mathcal{F}$. The fibre over $x \in M$ will often be denoted by F_x, $F_x := \pi^{-1}(x)$. Let $l_x : F_x \to \mathcal{F}$, $x \in M$, be the isomorphisms defined by the restricted decomposition functions; namely, as $\phi_W|_x : \{x\} \times \mathcal{F} \to F_x$, we define l_x via $\phi_W|_x(x, \psi) =: l_x^{-1}(\psi) \in \pi^{-1}(x)$ for every $\psi \in \mathcal{F}$. We call the maps l_x point-trivializing maps (isomorphisms).

Let $\langle\cdot|\cdot\rangle : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ be the (nondegenerate Hermitian) scalar product in the Hilbert space $\mathcal{F}$ and, respectively, for every $x \in M$ the map $\langle\cdot|\cdot\rangle_x : F_x \times F_x \to \mathbb{R}$ be the scalar product in the fibre F_x considered as a Hilbert space[8]. For a general Hilbert bundle (F, π, M), the scalar products $\langle\cdot|\cdot\rangle_x$, $x \in M$, and $\langle\cdot|\cdot\rangle$ are completely independent. Such a situation is unsatisfactory from the viewpoint of many applications for which the Hilbert spaces F_x, $x \in M$, and $\mathcal{F}$ are required to be *isometric*. We say that the vector structure of the Hilbert bundle (F, π, M) is *compatible* with its metric structure if the (linear) isomorphisms $l_x : F \to \mathcal{F}$ preserve the scalar products (are metric preserving), namely iff $\langle\varphi_x|\psi_x\rangle_x = \langle l_x(\varphi_x)|l_x(\psi_x)\rangle$ for every $\varphi_x, \psi_x \in F_x$. A Hilbert bundle with compatible vector and metric structure will be called a *compatible Hilbert bundle*. In such a bundle the linear isomorphisms $l_x|x \in M$ not only (isomorphically) connect the vector structures of the fibres F_x, $x \in M$, and $\mathcal{F}$, but they also transform the (Hermitian) metric structure $\langle\cdot|\cdot\rangle$ from $\mathcal{F}$ to F for every $x \in M$ according to

$$\langle\cdot|\cdot\rangle_x = \langle l_x \cdot |l_x \cdot\rangle \qquad x \in M \tag{3.1}$$

and, consequently, from F_x to $\mathcal{F}$ through

$$\langle\cdot|\cdot\rangle = \langle l_x^{-1} \cdot |l_x^{-1} \cdot\rangle_x \qquad x \in M. \tag{3.1$'$}$$

It is easy to see that the maps $l_{x\to y} := l_y^{-1} \circ l_x : \pi^{-1}(x) \to \pi^{-1}(y)$ are (i) fibre maps for fixed y, (ii) linear isomorphisms and (iii) isometric, i.e. metric preserving in a sense that

$$\langle l_{x\to y} \cdot |l_{x\to y} \cdot\rangle_y = \langle\cdot|\cdot\rangle_x. \tag{3.2}$$

[8] The map $x \mapsto \langle\cdot|\cdot\rangle_x$, $x \in M$, or the collection of maps $\{\langle\cdot|\cdot\rangle_x,\ x \in M\}$ is called a *fibre metric* on (F, π, M).

Rewording, we can say that $l_{x\to y}$ are fibre-isometric isomorphisms. Consequently, all of the fibres over the base and the standard fibre of a compatible Hilbert bundle are (linearly) isometric and isomorphic Hilbert spaces.

From now on in this investigation, *only compatible Hilbert bundles will be employed*. For the sake of brevity, we shall call them simply Hilbert bundles.

Now some definitions in compatible Hilbert bundles are in order. Notice that below we present the minimum of material concerning Hilbert bundles which is absolutely required for formulation of quantum mechanics in terms of bundles.

Defining the *Hermitian conjugate* map (operator) $A_x^\ddagger : \mathcal{F} \to F_x$ of a map $A_x : F_x \to \mathcal{F}$ by

$$\langle A_x^\ddagger \varphi | \chi_x \rangle_x := \langle \varphi | A_x \chi_x \rangle \qquad \varphi \in \mathcal{F} \qquad \chi_x \in F_x \tag{3.3}$$

we find (see (3.1))

$$A_x^\ddagger = l_x^{-1} \circ \left(A_x \circ l_x^{-1}\right)^\dagger \tag{3.4}$$

where the dagger denotes Hermitian conjugation in $\mathcal{F}$ (see (2.5)).

We call a map $A_x : F_x \to \mathcal{F}$ *unitary* if

$$A_x^\ddagger = A_x^{-1}. \tag{3.5}$$

Evidently, the isometric isomorphisms $l_x : F_x \to \mathcal{F}$ are unitary in this sense:

$$l_x^\ddagger = l_x^{-1}. \tag{3.6}$$

Similarly, the *Hermitian conjugate* map to a map $A_{x\to y} \in \{C_{x\to y} : F_x \to F_y,\ x, y \in M\}$ is a map $A_{x\to y}^\ddagger : F_x \to F_y$ defined via

$$\langle A_{x\to y}^\ddagger \Phi_x | \Psi_y \rangle_y := \langle \Phi_x | A_{y\to x} \Psi_y \rangle_x \qquad \Phi_x \in F_x \qquad \Psi_y \in F_y. \tag{3.7}$$

Its explicit form is

$$A_{x\to y}^\ddagger = l_y^{-1} \circ \left(l_x \circ A_{y\to x} \circ l_y^{-1}\right)^\dagger \circ l_x. \tag{3.8}$$

As $(\mathcal{A}^\dagger)^\dagger \equiv \mathcal{A}$ for any $\mathcal{A} : \mathcal{F} \to \mathcal{F}$, we have

$$\left(A_{x\to y}^\ddagger\right)^\ddagger = A_{x\to y}. \tag{3.9}$$

If $B_{x\to y} \in \{C_{x\to y} : F_x \to F_y,\ x, y \in M\}$, then a simple verification shows

$$\left(B_{y\to z} \circ A_{x\to y}\right)^\ddagger = A_{y\to z}^\ddagger \circ B_{x\to y}^\ddagger \qquad x, y, z \in M. \tag{3.10}$$

A map $A_{x\to y}$ is called *Hermitian* if

$$A_{x\to y}^\ddagger = A_{x\to y}. \tag{3.11}$$

A simple calculation proves that the maps $l_{x\to y} := l_y^{-1} \circ l_x$ are Hermitian.

A map $A_{x\to y} : F_x \to F_y$ is called *unitary* if it has a left inverse map and

$$A_{x\to y}^\ddagger = A_{y\to x}^{-1} \tag{3.12}$$

where $A_{x\to y}^{-1} : F_y \to F_x$ is the *left* inverse of $A_{x\to y}$, i.e. $A_{x\to y}^{-1} \circ A_{x\to y} := \mathrm{id}_{F_x}$.

A simple verification by means of (3.7) shows the equivalence of (3.12) with

$$\langle A_{y\to x} \cdot | A_{y\to x} \cdot \rangle_x = \langle \cdot | \cdot \rangle_y : F_y \times F_y \to \mathbb{C} \tag{3.12$'$}$$

i.e. the unitary maps are fibre-metric compatible in a sense that they preserve the fibre scalar (inner) product. Such maps will be called *fibre isometric* or simply *isometric*.

It is almost evident that the maps $l_{x\to y} = l_y^{-1} \circ l_x$ are unitary, that is we have[9]

$$l_{x\to y}^\ddagger = l_{x\to y} = l_{y\to x}^{-1} \qquad l_{x\to y} := l_y^{-1} \circ l_x : \pi^{-1}(x) \to \pi^{-1}(y). \tag{3.13}$$

[9] The Hermiticity and at the same time unitarity of $l_{x\to y}$ is not incidental as they define a (flat) linear transport (along paths or along the identity map of M) in (F, π, M) (see (3.23), the paragraph after (3.28) and footnote 12).

Let A be a morphism over M of (F, π, M), i.e. $A : F \to F$ and $\pi \circ A = \pi$, and $A_x := A|_{F_x}$. The *Hermitian conjugate* bundle morphism $A^\ddagger$ to A is defined by (cf (3.7))

$$\langle A^\ddagger \Phi_x | \Psi_x \rangle_x := \langle \Phi_x | A \Psi_x \rangle_x \qquad \Phi_x, \Psi_x \in F_x. \tag{3.14}$$

Thus (cf (3.8))

$$A_x^\ddagger := A^\ddagger\big|_{F_x} = l_x^{-1} \circ \left(l_x \circ A_x \circ l_x^{-1} \right)^\dagger \circ l_x. \tag{3.15}$$

A bundle morphism A is called *Hermitian* if $A_x^\ddagger = A_x$ for every $x \in M$, i.e. if

$$A^\ddagger = A \tag{3.16}$$

and it is called *unitary* if $A_x^\ddagger = A_x^{-1}$ for every $x \in M$, i.e. if

$$A^\ddagger = A^{-1}. \tag{3.17}$$

Using (3.14), we can establish the equivalence of (3.17) and

$$\langle A \cdot | A \cdot \rangle_x = \langle \cdot | \cdot \rangle_x : F_x \times F_x \to \mathbb{C}. \tag{3.17$'$}$$

Consequently the unitary morphisms are fibre-metric compatible; i.e., they are *isometric* in a sense that they preserve the fibre Hermitian scalar (inner) product.

Starting with the second part of this paper, we will need to deal with the differentiable properties of the employed Hilbert bundle (F, π, M). To make this possible, we will require the (total) bundle space F to be (at least) C^1 manifold[10]. Besides, we shall need the paths in M to have continuous tangent vectors; in our interpretation of M as a spacetime model, this corresponds to the existence of velocities of the (point-like) particles and observers. To ensure this natural requirement, we assume M to be a C^1 differentiable manifold[11]. Moreover, we shall need the point-trivializing isomorphisms l_x to have a C^1 dependence of $x \in M$, i.e. the mapping $l : F \to \mathcal{F}$ given by $l : u \mapsto l_{\pi(u)} u$ for $u \in F$ to be of class C^1 as a map between manifolds. A Hilbert bundle with the latter property will be called a C^1 bundle (or bundle of class C^1).

Let us summarize the basic requirements for the bundle (F, π, M) that will be employed in this paper: (i) it is a compatible Hilbert bundle; (ii) the bundle space F and the base M are C^1 differentiable manifolds; and (iii) it is of class C^1 and the set $\{l_x,\ x \in M\}$ of point trivializing (C^1 isometric) isomorphisms is fixed[12]. The isomorphisms l_x will frequently and explicitly be used throughout this paper. The formalism of the theory is not invariant under their choice but the corresponding transformation formulae are easily derivable and the physical predictions are independent of them. For instance, if $\{m_x\}$ is another set of point-trivializing isomorphisms, the scalar products $\langle \cdot | \cdot \rangle_x^l$ and $\langle \cdot | \cdot \rangle_x^m$ defined, respectively, by $\{l_x\}$ and $\{m_x\}$ are connected via the equality $\langle \cdot | \cdot \rangle_x^l = \langle \varphi_{l,m} \cdot | \varphi_{l,m} \cdot \rangle_x^m$ where the (fibre-preserving) bundle morphism $\varphi_{l,m} : F \to F$ is given by $\varphi_{l,m}|_{F_x} := \varphi_{l,m;x}; = m_x^{-1} \circ l_x : F_x \to F_x$. By means of the morphisms $\varphi_{l,m} = \varphi_{m,l}^{-1}$ the formalism can be transformed from a particular choice of $\{l_x\}$ to any other one $\{m_x\}$. Running some steps ahead, we have to say that the set $\{l_x\}$ cannot be fixed on the basis of conventional quantum mechanics; its particular choice is external to it. In this sense $\{l_x\}$ is a free parameter in the bundle formulation of quantum mechanics. Regardless of this, as we shall see, the predictions of the resulting theory are independent of the concrete choice of $\{l_x\}$ and coincide with those of conventional quantum mechanics.

[10] As the fibres $F_x \subset F$ are, generally, infinite dimensional, the dimension of F is generically infinity. The theory of such manifolds is given, for instance, in [50].

[11] In most applications M is supposed to be of class C^2 or C^3 (e.g. the Riemannian manifold of general relativity) or even C^∞ (e.g. the Minkowski spacetime of special relativity or the Euclidean space of classical/quantum mechanics).

[12] The last condition is equivalent on (F, π, M) to there being fixed a path-independent, of class C^1, and isometric linear transport along paths (*vide infra* section 3.2). From such a position, the formalism will be studied elsewhere. Here we notice that the maps $l_{x \to y}$ (see (3.13)) define such a transport: if $\gamma : J \to M$ and $s, t \in J$, by proposition 3.1 the map $l : \gamma \mapsto l^\gamma$ with $l^\gamma : (s, t) \mapsto l_{\gamma(s) \to \gamma(t)} = l_{\gamma(t)}^{-1} \circ l_{\gamma(s)}$ is a linear transport in (F, π, M); it is obviously path independent, of class C^1, and isometric as l_x are.

3.2. Linear transports along paths

The general theory of linear transports along paths in vector bundles is developed at length in [34, 35]. In the present investigation we shall need only a few definitions and results from these papers when the bundle considered is a Hilbert one (*vide supra* definition 3.1). The current section is devoted to their partial introduction and description.

Let (E, π, B) be a complex[13] vector bundle (see section 3.1 or, for example, [9, 53]) with bundle (total) space E, base B, projection $\pi : E \to B$ and isomorphic fibres $\pi^{-1}(x) \subset E$, $x \in B$. Let $\mathcal{E}$ be the (standard, typical) fibre of the bundle, i.e. a vector space to which all $\pi^{-1}(x)$, $x \in B$, are isomorphic. By J and $\gamma : J \to B$ we denote, respectively, a real interval and path in B.

Definition 3.2. *A linear transport along paths in the bundle (E, π, B) is a map L assigning to any path $\gamma : J \to B$ a map L^γ, a transport along γ, such that $L^\gamma : (s, t) \mapsto L^\gamma_{s\to t}$ where the map*

$$L^\gamma_{s\to t} : \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t)) \qquad s, t \in J \tag{3.18}$$

called a transport along γ from s to t, has the properties

$$L^\gamma_{s\to t} \circ L^\gamma_{r\to s} = L^\gamma_{r\to t} \qquad r, s, t \in J \tag{3.19}$$

$$L^\gamma_{s\to s} = \mathrm{id}_{\pi^{-1}(\gamma(s))} \qquad s \in J \tag{3.20}$$

$$L^\gamma_{s\to t}(\lambda u + \mu v) = \lambda L^\gamma_{s\to t} u + \mu L^\gamma_{s\to t} v \qquad \lambda, \mu \in \mathbb{C} \quad u, v \in \pi^{-1}(\gamma(s)) \tag{3.21}$$

where $\circ$ denotes composition of maps and id_X is the identity map of a set X.

Remark 3.1. Equations (3.19) and (3.20) mean that L is a *transport along paths* in the bundle (E, π, B) [46, definition 2.1], while (3.21) specifies that it is *linear* [46, equation (2.8)]. In the present paper only linear transports will be used.

This definition generalizes the concept of a parallel transport in the theory of (linear) connections (see [46, 56] and references therein for details and comparison).

A few comments on definition 3.2 are now in order. According to equation (3.18), a linear transport along paths may be considered as a path-dependent connection: it establishes a fibre (isomorphic—see below) correspondence between the fibres over the path along which it acts. By virtue of equation (3.21) this correspondence is linear. Such a condition is a natural one when vector bundles are involved; it simply represents a compatibility condition with the vectorial structure of the bundle (see [46, section 2.3] for details). Equation (3.20) is a formal realization of our intuitive and naive understanding that if we 'stand' at some point of a path without 'moving' along it, then 'nothing' should happen with the fibre over that point. This property fixes a 0-ary operation in the set of (linear) transports along paths, defining in it the 'unit' transport. Finally, the equality (3.19), which may be called a group property of the (linear) transports along paths, is a rigorous expression of the intuitive representation that the 'composition' of two (linear) transports along one and the same path must be a (linear) transport along the same path.

In general, different forms of (3.18)–(3.21) are well known properties of the parallel transports generated by (linear) connections (see [56]). For this reason these transports turn out to be special cases of the general (linear) transport along paths [56, theorem 3.1]. In particular, comparing definition 3.2 with [57, definition 2.1] and taking into account [57, proposition 4.1], we conclude that special types of linear transport along paths are the parallel

[13] All of our definitions and results also hold for real vector bundles. Most of them are valid for vector bundles over more general fields too but this is inessential for the following.

transport assigned to a linear connection (covariant differentiation) of the tensor algebra of a manifold [58, 59], Fermi–Walker transport [60, 61], Fermi transport [61], Truesdell transport [62, 63], Jaumann transport [64], Lie transport [59, 60], the modified Fermi–Walker and Frenet–Serret transports [65] etc. Consequently definition 3.2 is general enough to cover a list of important transports used in theoretical physics and mathematics. Thus studying the properties of the linear transports along paths, we can draw corresponding conclusions for any one of the transports mentioned[14].

From (3.19) and (3.20), we obtain that $L^{\gamma}_{s\to t}$ are invertible and

$$\left(L^{\gamma}_{s\to t}\right)^{-1} = L^{\gamma}_{t\to s} \qquad s,t\in J. \tag{3.22}$$

Hence the linear transports along paths are in fact linear isomorphisms between the fibres over the path along which they act.

The following two propositions establish the general structure of the linear transports along paths.

Proposition 3.1. *A map (3.18) is a linear transport along γ from s to t for every $s,t\in J$ iff there exists a vector space V isomorphic with $\pi^{-1}(x)$, $x\in B$ and a family of linear isomorphisms $\{F(s;\gamma):\pi^{-1}(\gamma(s))\to V,\ s\in J\}$ such that*

$$L^{\gamma}_{s\to t} = F^{-1}(t;\gamma)\circ F(s;\gamma) \qquad s,t\in J. \tag{3.23}$$

Proof. If (3.18) is a linear transport along γ from s to t, then, fixing some $s_0\in J$ and using (3.20) and (3.22), we obtain $L^{\gamma}_{s\to t} = L^{\gamma}_{s_0\to t}\circ L^{\gamma}_{s\to s_0} = \left(L^{\gamma}_{t\to s_0}\right)^{-1}\circ L^{\gamma}_{s\to s_0}$. So (3.23) holds for $V=\pi^{-1}(\gamma(s_0))$ and $F(s;\gamma)=L^{\gamma}_{s\to s_0}$. Conversely, if (3.23) is valid for some linear isomorphisms $F(s;\gamma)$, then a straightforward calculation shows that it converts (3.19) and (3.20) into identities and (3.21) holds due to the linearity of $F(s;\gamma)$. □

Proposition 3.2. *In a vector bundle (E,π,B), let there be given linear transport along paths with a representation (3.23) for some vector space V and linear isomorphisms $F(s;\gamma):\pi^{-1}(\gamma(s))\to V,\ s\in J$. Then for a vector space ${}^{\star}V$ there exist linear isomorphisms ${}^{\star}F(s;\gamma):\pi^{-1}(\gamma(s))\to {}^{\star}V,\ s\in J$ for which*

$$L^{\gamma}_{s\to t} = {}^{\star}F^{-1}(t;\gamma)\circ {}^{\star}F(s;\gamma) \qquad s,t\in J \tag{3.24}$$

iff there exists a linear isomorphism $D(\gamma):V\to {}^{\star}V$ such that

$$ {}^{\star}F(s;\gamma) = D(\gamma)\circ F(s;\gamma) \qquad s\in J. \tag{3.25}$$

Proof. If (3.25) holds, then the substitution of $F(s;\gamma)=\mathrm{D}^{-1}(\gamma)\circ{}^{\star}F(s;\gamma)$ into (3.23) results in (3.24). Vice versa, if (3.24) is valid, then from its comparison with (3.23) it follows that $D(\gamma) = {}^{\star}F(t;\gamma)\circ\left(F(t;\gamma)\right)^{-1} = {}^{\star}F(s;\gamma)\circ\left(F(s;\gamma)\right)^{-1}$ is the required (independent of $s,t\in J$) isomorphism. □

Let (E,π,B) be a vector bundle whose bundle space E is a C^1 differentiable manifold. A linear transport L^{γ} along $\gamma:J\to B$ is called *differentiable of class C^k*, $k=0,1$, or simply C^k transport, if for arbitrary $s\in J$ and $u\in\pi^{-1}(\gamma(s))$ the path $\overline{\gamma}_{s;u}:J\to E$ with $\overline{\gamma}_{s;u}(t):=L^{\gamma}_{s\to t}u\in\pi^{-1}(\gamma(t))$, $t\in J$, is a C^k mapping in the bundle space[15] E. If a C^k linear transport has a representation (3.23), the mapping $s\mapsto F(s;\gamma)$ is of class C^k. So, the transport L^{γ} is of class C^k iff $L^{\gamma}_{s\to t}$ has C^k dependence on s and t simultaneously.

[14] The concept of linear transport along paths in vector bundles can be generalized to that of transports along paths in arbitrary bundles [46] and to transports along maps in bundles [66]. An interesting consideration of the concept of (parallel) 'transport' (along closed paths) in connection with homotopy theory and the classification problem of bundles can be found in [67]. These generalizations will not be used in this paper.

[15] If E is of class C^r with $r=0,1,\ldots,\infty,\omega$, we can define in an obvious way a C^k transport for every $k\leqslant r$.

If $\{e_i(\cdot;\gamma)|i = 1,\dots,\dim\pi^{-1}(\gamma(s))\}$ is a C^k frame along γ, i.e. $\{e_i(s;\gamma)\}$ is a basis in $\pi^{-1}(\gamma(s))$ and the mapping $s \mapsto e_i(s;\gamma)$ is of class C^k for every i, L^γ is of class C^k iff its matrix $\boldsymbol{L}(t,s;\gamma)$ with respect to $\{e_i(s;\gamma)\}$, $s \in J$ has C^k dependence on s and t. Here the elements of $\boldsymbol{L}(t,s;\gamma)$ are defined via the expansion

$$L^\gamma_{s\to t}\big(e_i(s;\gamma)\big) =: L^j{}_i(t,s;\gamma)e_j(t;\gamma) \qquad s,t \in J. \tag{3.26}$$

A transport L along paths in (E,π,B), E being C^1 manifold, is said to be of class C^k, $k = 0,1$, if the corresponding transport L^γ along γ is of class C^k for every C^1 path $\gamma : J \to B$. Further we shall consider only C^1 linear transports along paths whose matrices will be referred to smooth frames along paths.

The above definition and results for linear transports along paths deal with the general case concerning arbitrary vector bundles and are therefore insensitive to the dimensionality of the bundle's base or fibres. Below we point out some peculiarities of the case of a Hilbert bundle whose fibres are generally infinite dimensional.

For linear transports in a Hilbert bundle all results of [34,35,46] are valid with a possible exception of those in which (local) bases in the fibres are involved. The cause for this is that the dimension of a Hilbert space is (generally) infinity, so there arise problems connected with the convergence or divergence of the corresponding sums or integrals. Below we try to avoid these problems and to formulate our assertions and results in an invariant way.

Of course, propositions 3.1 and 3.2 remain valid on Hilbert bundles; the only addition is that the vector spaces V and $^\star V$ are now Hilbert spaces.

In [34, section 3] are introduced the so-called *normal* frames for a linear transport along paths as a (local) field of bases in which (on some set) the matrix of the transport is the unit matrix. Further in this series [68], we shall see that the normal frames realize the Heisenberg picture of motion in the Hilbert bundle formulation of quantum mechanics.

Now (see below the paragraph after equation (3.28)) we shall establish a result specific for the Hilbert bundles that has no analogue in the general theory: a transport along paths is Hermitian if it is unitary. This assertion is implicitly contained in [45, section 3] (see the paragraph after equation (3.6)).

We call a (possibly linear) transport along paths in (F,π,M) *Hermitian* or *unitary* if it satisfies, respectively, (3.11) or (3.12) in which x and y are replaced with arbitrary values of the parameter of the transportation path, i.e. if respectively

$$\big(L^\gamma_{s\to t}\big)^\ddagger = L^\gamma_{s\to t} \qquad s,t \in J \quad \gamma : J \to M \tag{3.27}$$

$$\big(L^\gamma_{s\to t}\big)^\ddagger = \big(L^\gamma_{t\to s}\big)^{-1}. \tag{3.28}$$

A simple corollary from (3.22) is the equivalence of (3.27) and (3.28); therefore, a *transport along paths in a Hilbert bundle is Hermitian if it is unitary*, i.e. these concepts are equivalent. For such transports we say that they are *consistent* or *compatible* with the Hermitian structure (metric (inner product)) of the Hilbert bundle [69]. Evidently, they are *isometric* fibre maps along the paths where they act. Therefore, a transport along paths in a Hilbert bundle is isometric iff it is Hermitian or iff it is unitary[16].

3.3. Liftings of paths, sections and derivations along paths

A *lifting*[17] (in a vector bundle (E,π,B)) of a map $g : X \to B$, X being a set, is a map $\overline{g} : X \to E$ such that $\pi \circ \overline{g} = g$; in particular, the liftings of the identity map

[16] The author thanks Professor James Stasheff (Math-UNC, Chapel Hill, NC, USA) for suggesting in July 1998 the term 'isometric transport' in the context given.

[17] For details see, for example, [55].

id_B of the base B are called *sections* and their set is $\mathrm{Sec}(E,\pi,B) := \{\sigma|\sigma : B \to E,\ \pi\circ\sigma = \mathsf{id}_B\}$. Let $\mathrm{P}(A) := \{\gamma|\gamma : J \to A\}$ be the *set of paths* in a set A and $\mathrm{PLift}(E,\pi,B) := \{\lambda|\lambda : \mathrm{P}(B) \to \mathrm{P}(E),\ (\pi\circ\lambda)(\gamma) = \gamma \text{ for } \gamma \in \mathrm{P}(B)\}$ be the *set of liftings of paths* from[18] B to E. The set $\mathrm{PLift}(E,\pi,B)$ is (i) a natural $\mathbb{C}$-vector space if we put $(a\lambda + b\mu) : \gamma \mapsto a\lambda_\gamma + b\mu_\gamma$ for $a, b \in \mathbb{C}$, $\lambda, \mu \in \mathrm{PLift}(E,\pi,B)$ and $\gamma \in \mathrm{P}(B)$, where, for brevity, we write λ_γ for $\lambda(\gamma)$, $\lambda : \gamma \mapsto \lambda_\gamma$; (ii) a natural left module with respect to complex functions on B: if $f, g : B \to \mathbb{C}$, we define $(f\lambda + g\mu) : \gamma \mapsto (f\lambda)_\gamma + (g\mu)_\gamma$ with $(f\lambda)_\gamma(s) := f(\gamma(s))\lambda_\gamma(s)$ for $\gamma : J \to B$ and $s \in J$; (iii) a left module with respect to the set $\mathrm{PF}(B) := \{\varphi|\varphi : \gamma \mapsto \varphi_\gamma,\ \gamma : J \to B,\ \varphi_\gamma : J \to \mathbb{C}\}$ of *functions along paths* in the base B: for $\varphi, \psi \in \mathrm{PF}(B)$, we set $(\varphi\lambda + \psi\mu) : \gamma \mapsto (\varphi\lambda)_\gamma + (\psi\mu)_\gamma$ where $(\varphi\lambda)_\gamma(s) := (\varphi_\gamma\lambda_\gamma)(s) := \varphi_\gamma(s)\lambda_\gamma(s)$.

The dimension of $\mathrm{PLift}(E,\pi,B)$ as a $\mathbb{C}$-vector space is infinity but as a left $\mathrm{PF}(B)$-module is equal to that of (E,π,B) (i.e. of its fibres). In the last case a basis in $\mathrm{PLift}(E,\pi,B)$ can be constructed as follows. For every $\gamma : J \to B$ and $s \in J$, choose a basis $\{e_i(s;\gamma)|i = 1,\ldots,\dim\pi^{-1}(\gamma(s))\}$ in $\pi^{-1}(\gamma(s))$; if E is a C^1 manifold, we suppose $e_i(s;\gamma)$ to have a C^1 dependence on s. Define $e_i \in \mathrm{PLift}(E,\pi,B)$ by $e_i : \gamma \mapsto e_i|_\gamma := e_i(\cdot;\gamma)$, i.e. $e_i|_\gamma : s \mapsto e_i|_\gamma(s) := e_i(s;\gamma)$. The set $\{e_i\}$ is a basis in $\mathrm{PLift}(E,\pi,B)$; i.e., for every $\lambda \in \mathrm{PLift}(E,\pi,B)$ there are $\lambda^i \in \mathrm{PF}(B)$ such that $\lambda = \sum_i \lambda^i e_i$ and $\{e_i\}$ are $\mathrm{PF}(B)$-linearly independent. Actually, for $\gamma : J \to B$ and $s \in J$, we have $\lambda_\gamma(s) \in \pi^{-1}(\gamma(s))$, so there exist numbers $\lambda^i_\gamma(s) \in \mathbb{C}$ such that $\lambda_\gamma(s) = \sum_i \lambda^i_\gamma(s)e_i(s;\gamma)$. Defining $\lambda^i \in \mathrm{PF}(B)$ by $\lambda^i : \gamma \mapsto \lambda^i_\gamma$ with $\lambda^i_\gamma : s \mapsto \lambda^i_\gamma(s)$, we obtain $\lambda = \sum_i \lambda^i e_i$; if $e_i(\cdot;\gamma)$ is of class C^1, so are λ^i_γ. The $\mathrm{PF}(B)$-linear independence of $\{e_i\}$ is an evident corollary of the $\mathbb{C}$-linear independence of $\{e_i(s;\gamma)\}$. As we notice above, if E is C^1 manifold, we choose e_i, i.e. $e_i|_\gamma$, to be C^1 and, consequently, the components λ^i, i.e. λ^i_γ, are of class C^1 too.

Let (E,π,B) be a vector bundle whose bundle space E is C^1 manifold. A lifting $\lambda \in \mathrm{PLift}(E,\pi,B)$ is said to be of class C^k, $k = 0, 1$, if in some (and hence in any) C^k frame in $\mathrm{PLift}(E,\pi,B)$ its components are of class C^k along any C^k path, i.e. λ is of class C^k if λ_γ is of class C^k for every C^k path γ. Analogously, $\varphi \in \mathrm{PF}(B)$ is of class C^k if φ_γ is of class C^k for a C^k path γ. Denote by $\mathrm{PLift}^k(E,\pi,B)$, $k = 0, 1$, the set of C^k liftings of paths from B to E and by $\mathrm{PF}^k(B)$, $k = 0, 1$, the set of C^k functions along paths in B. If also the base B is C^1 manifold, we denote by $\mathrm{Sec}^k(E,\pi,B)$ the set of C^k sections of the bundle (E,π,B).

Definition 3.3. *A derivation along paths in* (E,π,B) *or a derivation of liftings of paths in* (E,π,B) *is a map*

$$D : \mathrm{PLift}^1(E,\pi,B) \to \mathrm{PLift}^0(E,\pi,B) \tag{3.29a}$$

which is $\mathbb{C}$*-linear,*

$$D(a\lambda + b\mu) = aD(\lambda) + bD(\mu) \tag{3.30a}$$

for $a, b \in \mathbb{C}$ *and* $\lambda, \mu \in \mathrm{PLift}^1(E,\pi,B)$*, and the mapping*

$$D^\gamma_s : \mathrm{PLift}^1(E,\pi,B) \to \pi^{-1}(\gamma(s)) \tag{3.29b}$$

defined via $D^\gamma_s(\lambda) := \big((D(\lambda))(\gamma)\big)(s) = (D\lambda)_\gamma(s)$ *and called derivation along* $\gamma : J \to B$ *at* $s \in J$*, satisfies the 'Leibnitz rule':*

$$D^\gamma_s(f\lambda) = \frac{\mathrm{d}f_\gamma(s)}{\mathrm{d}s}\lambda_\gamma(s) + f_\gamma(s)D^\gamma_s(\lambda) \tag{3.30b}$$

[18] Every linear transport L along paths provides a lifting of paths: for every $\gamma : J \to B$ fix some $s \in J$ and $u \in \pi^{-1}(\gamma(s))$, the mapping $\gamma \mapsto \overline{\gamma}_{s;u}$ with $\overline{\gamma}_{s;u}(t) := L^\gamma_{s\to t}u$, $t \in J$ is a lifting of paths from B to E.

for every $f \in \mathrm{PF}^1(B)$. *The mapping*

$$D^\gamma : \mathrm{PLift}^1(E,\pi,B) \to \mathrm{P}\big(\pi^{-1}(\gamma(J))\big) \tag{3.29c}$$

defined by $D^\gamma(\lambda) := (D(\lambda))|_\gamma = (D\lambda)_\gamma$ *is called derivation along* γ.

Before continuing with the study of linear transports along paths, we want to say a few words on the links between sections (along paths) and liftings of paths.

The set $\mathrm{PSec}(E,\pi,B)$ of *sections along paths* of (E,π,B) consists of mappings $\boldsymbol{\sigma} : \gamma \mapsto \boldsymbol{\sigma}_\gamma$ assigning to every path $\gamma : J \to B$ a section $\boldsymbol{\sigma}_\gamma \in \mathrm{Sec}\big((E,\pi,B)|_{\gamma(J)}\big)$ of the bundle restricted on $\gamma(J)$. Every (ordinary) section $\sigma \in \mathrm{Sec}(E,\pi,B)$ generates a section $\boldsymbol{\sigma}$ along paths via $\boldsymbol{\sigma} : \gamma \mapsto \boldsymbol{\sigma}_\gamma := \sigma|_{\gamma(J)}$, i.e. $\boldsymbol{\sigma}_\gamma$ is simply the restriction of σ on $\gamma(J)$; hence $\boldsymbol{\sigma}_\alpha = \boldsymbol{\sigma}_\gamma$ for every path $\alpha : J_\alpha \to B$ with $\alpha(J_\alpha) = \gamma(J)$. Every $\boldsymbol{\sigma} \in \mathrm{PSec}(E,\pi,B)$ generates a lifting $\hat{\boldsymbol{\sigma}} \in \mathrm{PLift}(E,\pi,B)$ by $\hat{\boldsymbol{\sigma}} : \gamma \mapsto \hat{\boldsymbol{\sigma}}_\gamma := \boldsymbol{\sigma}_\gamma \circ \gamma$; in particular, the lifting $\hat{\sigma}$ associated with $\sigma \in \mathrm{Sec}(E,\pi,B)$ is given via $\hat{\sigma} : \gamma \mapsto \hat{\sigma}_\gamma = \sigma|_{\gamma(J)} \circ \gamma$.

Every derivation D along paths generates a map

$$\overline{D} : \mathrm{PSec}^1(E,\pi,B) \to \mathrm{PLift}^0(E,\pi,B)$$

which may be called a *derivation of* C^1 *sections along paths*, such that if $\boldsymbol{\sigma} \in \mathrm{PSec}^1(E,\pi,B)$, then $\overline{D} : \boldsymbol{\sigma} \mapsto \overline{D}\boldsymbol{\sigma} = \overline{D}(\boldsymbol{\sigma})$, where $\overline{D}\boldsymbol{\sigma} : \gamma \mapsto \overline{D}^\gamma\boldsymbol{\sigma}$ is a lifting of paths defined by $\overline{D}^\gamma\boldsymbol{\sigma} : s \mapsto (\overline{D}^\gamma\boldsymbol{\sigma})(s) := D_s^\gamma\hat{\boldsymbol{\sigma}}$ with $\hat{\boldsymbol{\sigma}}$ being the lifting generated by $\boldsymbol{\sigma}$; i.e., $\gamma \mapsto \hat{\boldsymbol{\sigma}}_\gamma := \boldsymbol{\sigma}_\gamma \circ \gamma$. Notice that if $\gamma : J \to B$ has self-intersection points and $x_0 \in \gamma(J)$ is such a point, the map $\gamma(J) \to \pi^{-1}(\gamma(J))$ given by $x \mapsto \{D_s^\gamma(\hat{\boldsymbol{\sigma}})|\gamma(s) = x,\ s \in J\}$, $x \in \gamma(J)$, is generally multiple valued at x_0 and, consequently, it is not a section of $(E,\pi,B)|_{\gamma(J)}$.

If B is a C^1 manifold and for some $\gamma : J \to B$ there exists a subinterval $J' \subseteq J$ on which the restricted path $\gamma|J : J' \to B$ is without self-intersections, i.e. $\gamma(s) \neq \gamma(t)$ for $s,t \in J'$ and $s \neq t$, we can define the *derivation along* γ *of sections* over $\gamma(J')$ as a map

$$\mathsf{D}^\gamma : \mathrm{Sec}^1\big((E,\pi,B)|_{\gamma(J')}\big) \to \mathrm{Sec}^0\big((E,\pi,B)|_{\gamma(J')}\big) \tag{3.31}$$

such that

$$(\mathsf{D}^\gamma\sigma)(x) := D_s^\gamma\hat{\sigma} \qquad \text{for} \quad x = \gamma(s) \tag{3.32}$$

where $s \in J'$ is unique for a given x and $\hat{\sigma} \in \mathrm{PLift}\big((E,\pi,B)|_{\gamma(J')}\big)$ is given by $\hat{\sigma} = \sigma|_{\gamma(J')} \circ \gamma|_{J'}$. Generally the map (3.31) defined by (3.32) is multiple valued at the points of self-intersection of γ, if any, as $(\mathsf{D}^\gamma\sigma)(x) := \{D_s^\gamma\hat{\sigma} : s \in J,\ \gamma(s) = x\}$. The so-defined map $\mathsf{D} : \gamma \mapsto \mathsf{D}^\gamma$ is called *section derivation along paths*. As we said, it is single valued only along paths without self-intersections.

Generally a section along paths or lifting of paths does not define a (single-value) section of the bundle as well as there not corresponding to a lifting along paths some (single-value) section along paths. The last case admits one important special exception: if a lifting λ is such that the lifted path λ_γ is an 'exact topological copy' of the underlying path $\gamma : J \to B$, i.e. if there exist $s,t \in J$, $s \neq t$, for which $\gamma(s) = \gamma(t)$, then $\lambda_\gamma(s) = \lambda_\gamma(t)$. Such a lifting λ generates a section $\overline{\lambda} \in \mathrm{PSec}(E,\pi,B)$ along paths given by $\overline{\lambda} : \gamma \mapsto \overline{\lambda}_\gamma$ with $\overline{\lambda} : \gamma(s) \mapsto \lambda_\gamma(s)$. In the general case, the mapping $\gamma(s) \mapsto \lambda_\gamma(s)$ for a lifting λ of paths is multiple valued at the points of self-intersection of $\gamma : J \to B$, if any; for injective path γ this map is a section of $(E,\pi,B)|_{\gamma(J)}$. Such mappings will be called *multiple-valued sections along paths*.

With every derivation D along paths in (E,π,B) can be associated a derivation $\tilde{D}$ along paths in $\mathrm{mor}_B(E,\pi,B)$. To this end every lifting $\mathrm{PLift}(\mathrm{mor}(E,\pi,B))$ should be regarded as a map

$$A : \mathrm{PLift}(E,\pi,B) \to \mathrm{PLift}(E,\pi,B) \tag{3.33}$$

such that, if $\lambda \in \text{PLift}(E, \pi, B)$, $\gamma : J \to B$ and $s \in J$, then

$$A : \lambda \mapsto A(\lambda) : \gamma \mapsto (A(\lambda))_\gamma := A_\gamma(\lambda_\gamma) \qquad A_\gamma(\lambda_\gamma) : s \mapsto A_\gamma(s)(\lambda_\gamma(s)). \tag{3.34}$$

For every derivation D along paths in (E, π, B), we define

$$\tilde{D} : \text{PLift}^1(\text{mor}_B(E, \pi, B)) \to \text{PLift}^0(\text{mor}_B(E, \pi, B)) \tag{3.35}$$

by

$$\tilde{D} : A \mapsto \tilde{D}(A) := D \circ A \tag{3.36}$$

where in the rhs of the last equality $A \in \text{PLift}^1(\text{mor}_B(E, \pi, B))$ is considered as a map (3.33) given by (3.34). Putting

$$\tilde{D}^\gamma(A) := D^\gamma \circ A \qquad \tilde{D}^\gamma_s(A) := D^\gamma_s \circ A \tag{3.37}$$

it is a trivial verification to show that the map $\tilde{D}$ is a derivation along paths in $\text{mor}_B(E, \pi, B)$. The map $\tilde{D}$ will be called *induced (from D) derivation along paths*.

Definition 3.4. *The derivation D along paths generated by a C^1 linear transport L along paths is a map of type (3.29a) assigning to every path $\gamma : J \to B$ a map D^γ, derivation along γ generated by L, such that $D^\gamma : s \mapsto D^\gamma_s$, $s \in J$, is a map (3.29b), called derivation along γ at s assigned to L, given via*

$$D^\gamma_s(\lambda) := \lim_{\varepsilon \to 0} \left\{ \frac{1}{\varepsilon} \left[L^\gamma_{s+\varepsilon \to s} \lambda_\gamma(s+\varepsilon) - \lambda_\gamma(s) \right] \right\} \tag{3.38}$$

for every lifting $\lambda \in \text{PLift}^1(E, \pi, B)$ with $\lambda : \gamma \mapsto \lambda_\gamma$.

Remark 3.2. The operator D^γ_s is an analogue of the covariant differentiation assigned to a linear connection; cf, for example, [70, p 139, equation (12)].

Remark 3.3. Notice that if γ has self-intersections and $x_0 \in \gamma(J)$ is such a point, the mapping $x \mapsto \pi^{-1}(x), x \in \gamma(J)$ given by $x \mapsto \{D^\gamma_s(\lambda) | \gamma(s) = x,\ s \in J\}$ is, generally, multiple valued at x_0.

Let L be a linear transport along paths in (E, π, B). For every path $\gamma : J \to B$ choose some $s_0 \in J$ and $u_0 \in \pi^{-1}(\gamma(s_0))$. The mapping

$$\overline{L} : \gamma \mapsto \overline{L}^\gamma_{s_0,u_0} \qquad \overline{L}^\gamma_{s_0,u_0} : J \to E \qquad \overline{L}^\gamma_{s_0,u_0} : t \mapsto \overline{L}^\gamma_{s_0,u_0}(t) := L^\gamma_{s_0 \to t} u_0 \tag{3.39}$$

is, evidently, a lifting of paths.

Definition 3.5. *The lifting of paths $\overline{L}$ from B to E in (E, π, B) defined via (3.39) is called the lifting (of paths) generated by the (linear) transport L.*

Equations (3.20) and (3.23), combined with (3.38), immediately imply

$$D^\gamma_t(\overline{L}) \equiv 0 \qquad t \in J \tag{3.40}$$

$$D^\gamma_s(a\lambda + b\mu) = aD^\gamma_s\lambda + bD^\gamma_s\mu \qquad a, b \in \mathbb{C} \quad \lambda, \mu \in \text{PLift}^1(E, \pi, B) \tag{3.41}$$

where $s_0 \in J$ and $u(s) = L^\gamma_{s_0 \to s} u_0$ are fixed. In other words, equation (3.40) means that the lifting $\overline{L}$ is constant along every path γ with respect to D.

Let $\{e_i(s;\gamma)\}$ be a field of smooth bases along $\gamma : J \to B$, $s \in J$. Combining the linearity of L with (3.26) and (3.38), we find the explicit local action of[19] D^γ_s:

$$D^\gamma_s \lambda = \sum_i \left[\frac{\mathrm{d}\lambda^i_\gamma(s)}{\mathrm{d}s} + \Gamma^i{}_j(s;\gamma)\lambda^j_\gamma(s) \right] e_i(s;\gamma). \tag{3.42}$$

[19] Here and below we suppose the existence of derivatives such as $\mathrm{d}\lambda^i_\gamma(s)/\mathrm{d}s$, namely $\lambda^i_\gamma : J \to \mathbb{C}$, to be a C^1 mapping. This, of course, imposes some smoothness conditions on γ which we assume to hold. Evidently, for the purpose γ must be at least continuous. Without going into details, we notice that the most natural requirement for γ, when B is a manifold, is to admit it to be a C^1 map.

Here the *(two-index) coefficients* $\Gamma^i{}_j$ of the linear transport L are defined by

$$\Gamma^i{}_j(s;\gamma) := \frac{\partial L^i{}_j(s,t;\gamma)}{\partial t}\bigg|_{t=s} = -\frac{\partial L^i{}_j(s,t;\gamma)}{\partial s}\bigg|_{t=s} \tag{3.43}$$

and, evidently, uniquely determine the generated by L derivation D along paths.

A trivial corollary of (3.41) and (3.42) is the assertion that the derivation along paths generated by a linear transport is actually a derivation along paths (see definition 3.3).

If the transport matrix $\boldsymbol{L}$ has a representation

$$\boldsymbol{L}(t,s;\gamma) = \boldsymbol{F}^{-1}(t;\gamma)\boldsymbol{F}(s;\gamma) \tag{3.44}$$

for some nondegenerate matrix-valued function $\boldsymbol{F}$, which is a corollary of (3.23), from (3.43), we obtain

$$\boldsymbol{\Gamma}(s;\gamma) := \left[\Gamma^i{}_j(s;\gamma)\right] = \frac{\partial \boldsymbol{L}(s,t;\gamma)}{\partial t}\bigg|_{t=s} = \boldsymbol{F}^{-1}(s;\gamma)\frac{\mathrm{d}\boldsymbol{F}(s;\gamma)}{\mathrm{d}s}. \tag{3.45}$$

From here and (3.43), we see that the change $\{e_i\} \to \{e'_i = \sum_j A^j_i e_j\}$ of the local bases along γ with a nondegenerate C^1 matrix $A := \left[A^j_i\right]$ implies

$$\boldsymbol{\Gamma}(s;\gamma) = \left[\Gamma^i{}_j(s;\gamma)\right] \mapsto \boldsymbol{\Gamma}'(s;\gamma) = \left[\Gamma'^i{}_j(s;\gamma)\right]$$

with

$$\boldsymbol{\Gamma}'(s;\gamma) = A^{-1}(s;\gamma)\boldsymbol{\Gamma}(s;\gamma)A(s;\gamma) + A^{-1}(s;\gamma)\frac{\mathrm{d}A\,(s;\gamma)}{\mathrm{d}s}. \tag{3.46}$$

It is a fundamental result [34,35] that there exists a bijective correspondence between linear transports along paths and derivations along paths: a linear transport generates derivation via (3.38) and, *vice versa*, for every derivation along paths there exists a unique transport generating it by (3.38). Locally this correspondence is established by the coincidence of the transport and derivation coefficients[20].

Every transport L along paths in a vector bundle (E,π,B) generates a linear transport ${}^{\circ}L$ along paths in the bundle $\mathrm{mor}_B(E,\pi,B)$ of point-restricted morphisms over B in (E,π,B). If $\gamma : J \to B$, explicitly we have [69, equations (3.9)–(3.12)] ${}^{\circ}L : \gamma \mapsto {}^{\circ}L^{\gamma} : (s,t) \mapsto {}^{\circ}L^{\gamma}_{s\to t}$ $s,t \in J$ with

$${}^{\circ}L^{\gamma}_{s\to t}(\varphi_{\gamma(s)}) := L^{\gamma}_{s\to t} \circ \varphi_{\gamma(s)} \circ L^{\gamma}_{t\to s} \in (\pi_0^B)^{-1}(\gamma(t)) = \{\psi|\psi : \pi^{-1}(\gamma(t)) \to \pi^{-1}(\gamma(t))\} \tag{3.47}$$

for every $\varphi_{\gamma(s)} : \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(s))$. The transport ${}^{\circ}L$ will be described as *associated* with L (in $\mathrm{mor}_B(E,\pi,B)$).

The derivation generated by ${}^{\circ}L$ along paths in $\mathrm{mor}_B(E,\pi,B)$ will be denoted by ${}^{\circ}D$ and called the *derivation associated with the derivation* D generated by L. Therefore, if $A \in \mathrm{PLift}^1(\mathrm{mor}_B(E,\pi,B))$, then

$${}^{\circ}D^{\gamma}_s(A) := \lim_{\varepsilon\to 0}\left\{\frac{1}{\varepsilon}\left[{}^{\circ}L^{\gamma}_{s+\varepsilon\to s}A_{\gamma}(s+\varepsilon) - A_{\gamma}(s)\right]\right\}. \tag{3.48}$$

If the lifting A_{γ} of $\gamma : J \to B$ in the bundle space E_0^B of $\mathrm{mor}_B(E,\pi,B)$ is *linear* and the matrix of A_{γ} in $\{e_i(s;\gamma)\}$ is $\boldsymbol{A}_{\gamma}(s)$, then from (3.42)–(3.45) one finds the explicit matrix of ${}^{\circ}D^{\gamma}_s A$ as

$$[{}^{\circ}D^{\gamma}_s A] = [\boldsymbol{\Gamma}(s;\gamma), \boldsymbol{A}_{\gamma}(s)]_- + \frac{\mathrm{d}\boldsymbol{A}_{\gamma}\,(s)}{\mathrm{d}s} \tag{3.49}$$

[20] The coefficients (components) of derivation D along paths are defined by $D^{\gamma}_s\hat{e}_i = \sum_j \Gamma^j{}_i(s;\gamma)e_j(s;\gamma)$, where $\hat{e} : \gamma \mapsto e(\cdot;\gamma) : s \mapsto e(s;\gamma)$.

where $[\cdot, \cdot]_-$ means the commutator of matrices and the equality $\boldsymbol{L}(s, s; \gamma) = \mathbb{I}$, $\mathbb{I}$ being the unit matrix, was used (see (3.44) or (3.20)).

Under some assumptions, the matrix of the induced derivative $(\tilde{D}_s^\gamma A)\lambda = D_s^\gamma(A(\lambda))$ is (see (3.37) and (3.42))

$$[(\tilde{D}_s^\gamma A)\lambda] = \frac{\mathrm{d}\boldsymbol{A}_\gamma(s)}{\mathrm{d}s}\boldsymbol{\lambda}_\gamma(s) + \boldsymbol{A}_\gamma(s)\frac{\mathrm{d}\boldsymbol{\lambda}_\gamma(s)}{\mathrm{d}s} + \boldsymbol{\Gamma}(s;\gamma)\boldsymbol{A}_\gamma(s)\boldsymbol{\lambda}_\gamma(s) \qquad (3.50)$$

where $\lambda \in \mathrm{PLift}^1(E, \pi, B)$ is a C^1 lifting along paths in (E, π, B) and $\boldsymbol{\lambda}_\gamma(s)$ is the matrix of $\lambda_\gamma(s)$ in $\{e_i(s;\gamma)\}$.

In our investigation the above-presented general definitions and results will be applied to the particular case of a Hilbert bundle. Since its dimension is generically infinite, some problems connected with convergence of sums (which generally are integrals) or decompositions such as $\sum_i \lambda^i e_i$ could arise. We shall comment on these problems in the second part of the paper.

4. The Hilbert bundle description of quantum mechanics

As we shall see in this investigation, the Hilbert bundles provide a natural mathematical framework for a geometrical formulation of quantum mechanics. In it all quantum mechanical quantities, such as Hamiltonians, observables and wavefunctions, have an adequate description. For instance, the evolution of a systems is described as an appropriate (parallel or, more precisely, linear) transport of the system's state liftings of paths or sections along paths. We have to emphasize on the fact that the new bundle formulation of quantum mechanics and the conventional one are completely equivalent at the present stage of the theory.

4.1. Brief literature overview

Several attempts have been made for a (partial) (re-) formulation of nonrelativistic quantum mechanics in terms of bundles. Works containing such material were mentioned in section 1. Below are marked only those which directly or indirectly lead to some essential elements of our approach to quantum mechanics.

It seems that an appropriate bundle approach to quantum mechanics was developed for the first time in [38], where the single Hilbert space of quantum mechanics is replaced with infinitely many copies of it forming a bundle space over the one-dimensional 'time' manifold (i.e. over $\mathbb{R}_+$). In this Hilbert fibre bundle the quantum evolution is (equivalently) described as a kind of 'parallel' transport of suitable objects over the bundle's base. A similar treatment of the quantum evolution as a parallel transport in a Hilbert bundle over time (manifold) is stated in [43].

Analogous construction, a Hilbert bundle over the system's phase space, is used in Prugovečki's approach to quantum theory (see e.g. the references in [32]).

The gauge, i.e. linear connection, structure in quantum mechanics is first mentioned in [19]. That structure is pointed out to be connected with the system's Hamiltonian. This observation will find a natural explanation in our work (see [71]).

Some ideas concerning the interpretation of quantum evolution as a kind of a 'parallel' transport in a Hilbert bundle can also be found in [36, 39].

4.2. Motivation

Below are presented some nonexactly rigorous ideas and statements whose only purpose is the *motivation* for applying the fibre bundle formalism to quantum mechanics. Other excellent arguments and motives confirming this approach are given in [38, 43, 44], to which papers the reader is referred for details. Arguments in favour of the bundle approach to quantum mechanics will be also obtained *a posteriori* after its consistent development.

Let M be a differentiable manifold, representing in our context the space in which the (nonrelativistic) quantum mechanical objects 'live', i.e. the usual three-dimensional coordinate space (isomorphic to $\mathbb{R}^3$ with the corresponding structures)[21]. Let $\gamma : J \to M$, J being an $\mathbb{R}$-interval, be the trajectory of an observer describing the behaviour of a quantum system at any moment $t \in J$ by a state vector $\Psi_\gamma(t)$ depending on t and, possibly, on[22] γ. For a fixed point $x = \gamma(t) \in M$ the variety of state vectors describing a quantum system and corresponding to different observers form a Hilbert space $F_{\gamma(t)}$ which *depends on $\gamma(t) = x$, but not on γ and t separately*[23].

Remark 4.1. As we said above in footnote 21, the next considerations are completely valid mathematically if M is an arbitrary differentiable manifold and γ is a path in it. In this sense M and γ are free parameters in our theory and their concrete choice is subject only to *physical constraints*, first of all, ones requiring adequate physical interpretation of the resulting theory. (The arbitrariness of M in a similar construction is mentioned in [39, section I] too.) Typical candidates for M are the three-dimensional Euclidean space $\mathbb{E}^3$, $\mathbb{R}^3$, the four-dimensional Minkowski space M^4 of special relativity or the Riemannian space V_4 of general relativity, the system's configuration or phase space, the 'time' manifold $\mathbb{R}_+ := \{a : a \in \mathbb{R},\ a > 0\}$ etc. Correspondingly, γ obtains interpretation as a particle's trajectory, its world line and so on. The degenerate case when M consists of a single point corresponds (up to an isomorphism—see below) to the conventional quantum mechanics. Throughout this paper, we most often take $M = \mathbb{R}^3$ as a natural choice corresponding to the nonrelativistic case investigated here, but, as we said, this is not required by necessity. Elsewhere we shall see that $M = M^4$ or V_4 are natural choices in the relativistic region. An expanded comment on these problems will be given in the concluding part of this series. Here we want only to note that the interpretation of γ as an observer's (particle's) trajectory or world line, as accepted in this paper, is a reasonable but not necessary one. Maybe a more adequate one is to interpret γ as a mean (in the quantum mechanical sense) trajectory of some point particle, but this does not change anything in the mathematical structure of the bundle approach proposed here.

The spaces $F_{\gamma(t)}$ must be isomorphic as, from a physical viewpoint, they simply represent the possible variety of state vectors from different positions. In this way over M there arises a natural bundle structure, namely a *Hilbert bundle* (F, π, M) with a total space F, projection $\pi : F \to M$ and isomorphic fibres $\pi^{-1}(x) := F_x$. Since F_x, $x \in M$, are isomorphic, there exists a Hilbert space $\mathcal{F}$ and (linear) isomorphisms $l_x : F_x \to \mathcal{F}$, $x \in M$. Mathematically $\mathcal{F}$ is the typical (standard) fibre of (F, π, M). The maps $\Psi_\gamma : J \to \pi^{-1}(\gamma(J))$ can be considered as sections over any part of γ without self-intersections (see below).

Here are other similar arguments leading to the same Hilbert bundle (F, π, M) with fibre $\mathcal{F}$. Suppose $M = \mathbb{E}^3$ is the three-dimensional Euclidean space (-time) model of classical/quantum mechanics as above. Let O and O′ be two observers in $\mathbb{E}^3$ connected via special Galilean

[21] In the following M can naturally be considered also as the Minkowski spacetime of special relativity. In this case the below-defined observer's trajectory γ is his world line. However, we avoid this interpretation because only the nonrelativistic case is investigated here. It is important to note that mathematically all of what follows is valid in the case when by M is understood an arbitrary differentiable manifold. The physical interpretation of these cases will be given elsewhere.

[22] In this way we introduce the (possible) explicit dependence of the description of system's state on the concrete observer with respect to which it is determined.

[23] If there exists a global time, as in the nonrelativistic quantum mechanics, the parameter $t \in J$ can be taken as such. Otherwise by t we have to understand the local ('proper' or 'eigen-') time of a concrete observer.

transformation with parameter $\vec{v}$, i.e. O′ moves with constant velocity $\vec{v}$ with respect to O. This implies that the radius vectors $\vec{r}$ and $\vec{r}\,'$ of an arbitrary point in $\mathbb{E}^3$ with respect to O and O′ are related by $\vec{r}\,' = \vec{r} - \vec{v}t$, where $t \in \mathbb{R}$ is the (absolute Newtonian) time. It is known [72–74] that the Schrödinger equation is *not* invariant under the (special) Galilean group in the following sense. If $\psi(\vec{r}, t)$ is the wavefunction of a system with respect to O, then the Galilean transformation $(\vec{r}, t) \mapsto (\vec{r'} = \vec{r} - \vec{v}t, t' = t)$ implies the change $\psi(\vec{r}, t) \mapsto \psi'(\vec{r}\,', t')$, where the new vector $\psi'(\vec{r}\,', t')$ belongs to the same Hilbert space as $\psi(\vec{r}, t)$ but it does not satisfy the Schrödinger equation corresponding to the appropriately changed Hamiltonian[24]. However, one can revert the situation: if $\Psi(\vec{r}\,', t')$ is the wavefunction of the same system relative to O′, the inverse Galilei transformation $(\vec{r}\,', t') \mapsto (\vec{r} = \vec{r}\,' + \vec{v}t', t = t')$ leads to a vector $\varphi(\vec{r}, t)$ which is not a solution of the Schrödinger equation with respect to O. Since $\psi(\vec{r}, t)$ and $\Psi(\vec{r}\,', t')$ are the wavefunctions of one and the same system (relative to O and O′ respectively), they should be uniquely expressible through each other. This implies that if $\mathcal{F}$ and F_t are the Hilbert spaces of the system's states with respect to O and O′ respectively[25], they must be isomorphic. Therefore along the path $\vec{\gamma} : t \mapsto \vec{\gamma}(t) := \vec{v}t \in \mathbb{E}^3$ (with respect to O) there arises a family $\{F_t : t \in \mathbb{E}\}$ of isomorphic Hilbert space such that $F_t|_{t=0} = \mathcal{F}$ is the system's Hilbert space relative to O. Since the vector $\vec{v}$ (i.e. the observer O′) is completely arbitrary, from here it follows that to every point $x \in \mathbb{E}^3$ is assigned a Hilbert space F_x which is isomorphic to $\mathcal{F}$. In this way over $M = \mathbb{E}^3$ there arises a Hilbert bundle (F, π, M) with bundle space $F = \cup_{x \in M} F_x$, projection $\pi : F \to M$ with $\pi^{-1}(x) := F_x$ for all $x \in M$ and standard fibre $\mathcal{F}$, which is the system's Hilbert space with respect to the initial observer O. As F_x and $\mathcal{F}$ are isomorphic for any $x \in M$, there are isomorphisms $l_x : F \to \mathcal{F}$. In particular, for $x = \vec{\gamma}(t) = \vec{v}t$ the isomorphism $l_{\vec{v}t} : F_{\vec{v}t} \to \mathcal{F}$ realizes the correspondence $\Psi(\vec{r}\,', t') \mapsto \psi(\vec{r}, t)$ between the wavefunctions of the considered system with respect to the observers O and O′ connected via a special Galilei transformation. Summarizing the above, we can say that the Galilei invariance (of the classical space (-time) model) naturally leads to a Hilbert bundle description of the solutions of the Schrödinger equation. Such a description reflects the Galilean invariance in a sense that, given the wavefunction $\psi(\vec{r}, t) \in F_0 = \mathcal{F}$ of a system with respect to some observer O, one can immediately obtain the wavefunction $\Psi(\vec{r}\,', t') \in F_t$ of the same system with respect to any observer O′ moving relative to O with constant velocity $\vec{v}$ by means of the isomorphisms $l_x, x \in M$.

Now a natural question arises: how is the quantum evolution in time in the bundle constructed described? There are two almost 'evident' ways to do this. On one hand, we can postulate the conventional quantum mechanics in every fibre F_x, i.e. the Schrödinger equation for the state vector $\Psi_\gamma(t) \in F_{\gamma(t)}$ with $F_{\gamma(t)}$ being (an isomorphic copy of) the system's Hilbert space, but the only thing one obtains in this way is an isomorphic image of the usual quantum mechanics in any fibre over M. Therefore, one cannot expect new results or descriptions in this direction (see below (4.2) and the comments after it). On the other hand, we can demand the ordinary quantum mechanics to be valid in the fibre $\mathcal{F}$ of the bundle (F, π, M). This means identifying $\mathcal{F}$ with the system's Hilbert space of states and describing the quantum time evolution of the system via the vector

$$\psi(t) = l_{\gamma(t)}(\Psi_\gamma(t)) \in \mathcal{F} \tag{4.1}$$

which evolves according to (2.1) or (2.6). This approach is accepted in the present investigation. What we intend to do further, is, by using the basic relation (4.1), to 'transfer' the quantum

[24] For the explicit form of $\psi'(\vec{r}\,', t')$, see [5, the exercise to section 17]. For representations of the Galilei group preserving the solutions of the Schrödinger equation, see [73] where it is also explicitly proved that ψ and ψ' cannot satisfy the Schrödinger equation simultaneously.

[25] Since O′ depends on t, we write F_t instead of F'.

mechanics from $\mathcal{F}$ to (F, π, M) or, in other words, to investigate the quantum evolution in terms of the vector $\Psi_\gamma(t)$ connected with $\psi(t)$ via (4.1). Since $l_x,\ x \in M$ are isomorphisms, both descriptions are *completely equivalent*. This equivalence resolves a psychological problem that may arise *prima facie*: the single Hilbert space $\mathcal{F}$ of standard quantum theory [1–5] is replaced with, generally, an infinite number of copies $F_x,\ x \in M$, thereof (cf [38]). In the present investigation we shall show that the merit one gains from this is an entirely geometrical reformulation of quantum mechanics in terms of Hilbert fibre bundles.

The evolution of a quantum system will be described in a fibre bundle (F, π, M) with fixed isomorphisms $\{l_x,\ x \in M\}$ such that $l_x : F_x \to \mathcal{F}$, where $\mathcal{F}$ is the Hilbert space in which the system's evolution is described through the usual Schrödinger picture of motion.

In the Schrödinger picture a quantum system is described by a state vector ψ in $\mathcal{F}$. Generally [47] ψ depends (maybe implicitly) on the observer with respect to which the evolution is studied[26] and it satisfies the Schrödinger equation (2.6). We shall refer to this representation of quantum mechanics as a *Hilbert space description*. In the new *(Hilbert fibre) bundle description*, which will be studied below, the linear isomorphisms $l_x : F_x = \pi^{-1}(x) \to \mathcal{F},\ x \in M$ are supposed to be arbitrarily fixed[27] and the quantum systems are described by *state liftings of paths or sections along paths* Ψ of a bundle (F, π, M) whose typical fibre is the Hilbert space $\mathcal{F}$ (the same Hilbert space as in the Hilbert space description).

Generally, to any vector $\varphi \in \mathcal{F}$ there corresponds a unique (global) section $\overline{\Phi} \in \mathrm{Sec}(F, \pi, M)$ defined via

$$\overline{\Phi} : x \mapsto \overline{\Phi}_x := l_x^{-1}(\varphi) \in F_x \qquad x \in M \qquad \varphi \in \mathcal{F}. \tag{4.2}$$

Consequently to a state vector $\psi(t) \in \mathcal{F}$ one can assign the (global) section $\overline{\Psi}(t)$, $\overline{\Psi}(t) : x \mapsto \overline{\Psi}_x(t) = l_x^{-1}(\psi(t)) \in F_x$ and thus obtain in F_x for every $x \in M$ an isomorphic picture of (the evolution in) $\mathcal{F}$, but in this way one cannot expect significantly new results as the evolution in $\mathcal{F}$ is simply replaced with the evolution (linearly isomorphic to it) in F_x for every arbitrary fixed[28] $x \in M$. This reflects the fact that the quantum mechanical description is defined up to linear isomorphism(s) (see note 4.4 below). Additionally, in contrast to the bundle description, in this way one loses the explicit dependence on the observer, so in it one cannot obtain really new results with respect to the Hilbert space description.

4.3. Basic ideas and statement of the problems

Taking into account the (more or less heuristic) arguments from the previous subsection, we pose the following problem: given a Hilbert bundle (F, π, M) with the properties described in section 3.1 and a path $\gamma : J \to M$, describe the quantum evolution of some quantum system in this bundle provided the standard fibre $\mathcal{F}$ is the system's Hilbert space of states. For the moment, we identify the bundle's base M with the spacetime model used: for definiteness we take for it the three-dimensional Euclidean space $\mathbb{E}^3$. The path mentioned will be interpreted as a trajectory of a certain observer; correspondingly its parameter will be treated as a (global) time.

At precisely this point some natural questions arise. First of all, why should one replace the single conventional Hilbert space of the system with a Hilbert bundle, i.e. with a (generally) infinite number of different 'local' Hilbert spaces, each of which is associated with a single space point? Moreover, why is the introduced reference path required? These are basic

[26] Usually this dependence is not written explicitly, but it is always present as actually t is the time with respect to a given observer.
[27] The particular choice of $\{l_x\}$ (and, consequently, of the fibres F_x) is inessential for our investigation.
[28] The machinery of global sections such as (4.2) is used in [39] for the bundle approach to quantum mechanics contained in this paper.

moments which *a posteriori* will be justified by the results but *a priori* their essence is in the following. Conventionally, the system's evolution is described by *different state vectors, one for every instant of time, in the unique Hilbert space of the system*. These state vectors *generically depend on the observer* with respect to which the system is explored. This dependence is often an implicit one in quantum mechanics but it is always present: the states might be different because, for example, the observers could have different velocities, or be rotated relative to each other. So, different observers assign, generally, different state vectors to one and the same quantum system at a given moment and *these vectors belong to the (initial) Hilbert space of the system*. In this context the shift to a Hilbert bundle pursues a twofold goal: the *explicit observer dependence* of the 'state vectors'[29] and the *split of the time values of the 'state vectors' into different Hilbert spaces*. We achieve this by describing the system's state at a time $t \in J$ with respect to the observer with trajectory $\gamma : J \to M$ with a 'state vector' from the 'local' Hilbert space attached to the point $\gamma(t)$, i.e. *from the fibre* $F_{\gamma(s)} := \pi^{-1}(\gamma(s))$. Besides, through γ, the observer dependence of the 'state vectors' is introduced, maybe implicitly, via the Hamiltonian, which does not exists *per se* but is always given with respect to some concrete observer. Consequently, if we have two observers with trajectories $\alpha : J_\alpha \to M$ and $\beta : J_\beta \to M$ with $J_\alpha \cap J_\beta \neq \varnothing$, at a moment $t \in J_\alpha \cap J_\beta$ they will describe the state of a system via some vectors $\Psi_\alpha \in F_{\alpha(t)}$ and $\Psi_\beta \in F_{\beta(t)}$. In particular, if it happens that at the moment t the observers are at one and the same point $x = \alpha(t) = \beta(t)$, the 'state vectors' Ψ_α and Ψ_β will be from a single fibre, that over x, i.e. the Hilbert space $F_x = \pi^{-1}(x)$, but generally these vectors will be different unless the observers are absolutely identical at the moment[30] t.

At the moment it is not clear what one gains from 'unwrapping' the time evolution from the single Hilbert space $\mathcal{F}$ to a collection $\{F_{\gamma(t)} | t \in J\}$ of 'local' Hilbert spaces along the observer's trajectory γ. We shall try to explain this in section 4.4. In advance, we want only to state the main merit of the proposed approach: a self-consistent purely geometrical formulation of (non-) relativistic quantum mechanics in terms of Hilbert bundles.

Now it is time for explicit rigorous statement of the basic assertions and the problems we are going to solve later in this investigation. Note that if the opposite is not explicitly stated, we consider only pure quantum states that conventionally are described by vectors in a Hilbert space.

Postulate 4.1. *Let there be given a quantum system and let $\mathcal{F}$ be its Hilbert space of states. To this system we assign a C^1 compatible Hilbert bundle (F, π, M) with bundle space F, projection $\pi : F \to M$ and base M. In addition, we suppose:*

(i) The base M and the bundle space F are C^1 differentiable manifolds.
(ii) The point-trivializing (isometric) isomorphisms $l_x : \pi^{-1}(x) \to \mathcal{F}$, $x \in M$, are fixed and of class C^1. Their dependence on x is also required to be of class C^1, i.e. (F, π, M) is of class C^1.
(iii) The (standard) fibre of (F, π, M) is the system's Hilbert space of states $\mathcal{F}$ in which the conventional quantum mechanics is valid.

Note 4.1. It should be emphasized that here we introduce two parameters which are left free from the quantum mechanics and are external to it: the base M and the set of isomorphisms $\{l_x\}$. For the sake of physical interpretation (see remark 4.1), we identify M with the space (-time) model, in particular with the three-dimensional Euclidean space $\mathbb{E}^3$ (or the Minkowski

[29] Here we use inverted commas as, actually, the right term is bundle state vector, i.e. a state lifting or section at some point; *vide infra* in this section.
[30] For example, if the observers have nonzero relative acceleration at x, it is quite natural that they will assign different 'state vectors' to the system at the moment t.

four-dimensional spacetime M^4, or the Riemannian four-manifold V_4 of general relativity etc). This does not influence the basic scheme which is valid for arbitrary manifold M. As concerns the set $\{l_x\}$, in this paper we consider it as given and its analysis and interpretation will be given elsewhere. In this connection, we want to note three things: (i) the arbitrariness in $\{l_x\}$ reflects the natural one in the choice of the system's Hilbert space of states which is defined up to isomorphism; (ii) as we shall see, the mathematical formalism depends on the choice of $\{l_x\}$ but the physically predictable results (the mean values (mathematical expectations) of the operators) do not; (iii) in another investigation we intend to show that on the basis of the set $\{l_x\}$ of isomorphisms there is very likely to be achieved a kind of unification of quantum mechanics and gravity.

Definition 4.1. *The bundle (F, π, M) introduced via postulate 4.1 will be called the Hilbert bundle (of states) of the quantum system, or simply the system's Hilbert bundle (of states).*

Postulate 4.2. *Let $J \subseteq \mathbb{R}$ be the real interval representing the period of time in which a quantum system is investigated, (F, π, M) be its Hilbert bundle and $\gamma : J \to M$ be a C^1 path in the base M. In (F, π, M) the state of the system at a moment $t \in J$ is described by a map Ψ assigning to a pair (γ, t) a vector $\Psi_\gamma(t) \in \pi^{-1}(\gamma(t)) = F_{\gamma(t)}$ such that*

$$\Psi_\gamma(t) = l^{-1}_{\gamma(t)}(\psi(t)) \in F_{\gamma(t)} \tag{4.3}$$

where $\psi(t) \in \mathcal{F}$ is the conventional state vector in the system's Hilbert space of states ($\equiv$ the bundle's fibre) describing the system's state at the moment t in the (usual) quantum mechanics.

Definition 4.2. *The description of a quantum system via the map Ψ (ψ) in the Hilbert bundle (F, π, M) (Hilbert space $\mathcal{F}$) will be called the Hilbert bundle (Hilbert space) description (of the quantum mechanics of the system).*

Note 4.2. Since the maps $l_x : F \to \mathcal{F}$ are isomorphisms, the description of quantum states by Ψ and ψ is completely equivalent.

Note 4.3. As we said above, the path γ will be physically interpreted as a trajectory (or, possibly, world line) of an observer moving in M and with respect to which the quantum system is studied (or 'who' investigates it). So, in the bundle description the 'state vector' $\Psi_\gamma(t)$, representing the system state at a moment t, explicitly depends on the observer which is depicted in the index γ in $\Psi_\gamma(t)$. This is contrary to the conventional quantum mechanics, where this dependence is implicitly assumed almost everywhere. Thus we come to the above-mentioned situation: different observers describe the system's state at a fixed moment by vectors from, generally, different fibres of the bundle; these vectors belong to one and the same fibre over some point in M iff the observers happen to be simultaneously in it but, even in this case, the vectors need not to coincide, they are generically different unless the observers are absolutely identical.

Note 4.4. The bundle, as well as the conventional, description of quantum mechanics is defined up to linear isomorphism(s). In fact, if $\imath : \mathcal{F} \to \mathcal{F}'$, $\mathcal{F}'$ being a Hilbert space, is a linear isomorphism (which may depend on the time t), then $\psi'(t) = \imath(\psi(t))$ equivalently describes the evolution of the quantum system in $\mathcal{F}'$. (Note that in this way, for $\mathcal{F}' = \mathcal{F}$, one can obtain the known pictures of motion in quantum mechanics—see [3].) In the bundle case the shift from $\mathcal{F}$ to $\mathcal{F}'$ is described by the transformation $l_x \to l'_x := \imath \circ l_x$, which reflects the arbitrariness in the choice of the typical fibre (now $\mathcal{F}'$ instead of $\mathcal{F}$) of (F, π, M). There is also arbitrariness in the choice of the fibres $F_x = \pi^{-1}(x)$, which is of the same character as that in the case of $\mathcal{F}$; namely, if $\imath_x : F_x \to F'_x$, $x \in M$ are linear isomorphisms, then the fibre bundle (F', π', M) with $F' := \bigcup_{x\in M} F'_x$, $\pi'|_{F'_x} := \pi \circ \imath_x^{-1}$, typical fibre $\mathcal{F}$, and isomorphisms

$l'_x := l_x \circ \iota_x^{-1}$ can equivalently be used to describe the state of a quantum system. In the most general case, we have a fibre bundle (F', π', M) with fibres $F'_x = \iota_x^{-1}(F_x)$, typical fibre $\mathcal{F}' = \iota(\mathcal{F})$ and isomorphisms $l'_x := \iota \circ l_x \circ \iota_x^{-1} : F'_x \to \mathcal{F}'$. Further we will not be interested in such generalizations. Thus, we shall suppose that all of the mentioned isomorphisms are fixed.

Let us now look at the mathematical nature of the map Ψ introduce via postulate 4.2. On one hand, as the notation suggests, the mapping $\Psi : \gamma \mapsto \Psi_\gamma$ with $\Psi_\gamma : t \mapsto \Psi_\gamma(t)$ is a lifting of paths, $\Psi \in \mathrm{PLift}(F, \pi, M)$, which is a trivial corollary of (4.3). On the other hand, we can consider Ψ as a multiple-valued section along paths; to this end one has to put $\Psi : \gamma \mapsto {}_\gamma\Psi$, $\gamma : J \to M$, with ${}_\gamma\Psi : x \mapsto \{\Psi_\gamma(t) | \gamma(t) = x,\ t \in J\}$ for $x \in \gamma(J)$. If one employs multiple-valued sections along paths, the basic problem is how exactly the values corresponding to some 'time' value t are chosen and how the transition between different 'time' values is depicted; of course, this problem arises at the points of self-intersection of γ, if any. Mathematically the work with multiple-valued maps is considerably more difficult than the treatment of single-valued ones. The correct rigorous treatment of Ψ as a section requires additional rules describing, besides the correspondence $\gamma(t) \mapsto {}_\gamma\Psi(\gamma(t))$, the mapping $t \mapsto \Psi_\gamma(t)$, which is equivalent to the consideration of Ψ as a lifting of paths. For this reason, in the general case, we shall look on Ψ as a lifting of paths. There is one important special case when both approaches to Ψ are transparently equivalent: when only paths γ without self-intersections are employed. This is a consequence of the fact that now the map $\gamma : J \to \gamma(J)$ is bijective as $\gamma : J \to M$ is injective. In particular, if for given γ there is a subinterval $J' \subset J$ such that the restricted path $\gamma|_{J'}$ is injective, the maps $\Psi_{\gamma|J'}$ and ${}_{\gamma|J'}\Psi$ are completely equivalent representations of Ψ along $\gamma|J'$.

The physical preference to interpret Ψ as a section or lifting depends on the concrete choice of M and the corresponding interpretation of γ. For example, if M is the spacetime of special or general relativity and γ is the world line of (a real point-like) observer, then γ is without self-intersections and Ψ along γ can naturally be interpreted as a section in $\mathrm{Sec}(F, \pi, M)|_{\gamma(J)}$. On the other hand, if $M = \mathbb{E}^3$ is the Euclidean space of classical mechanics and $\gamma : J \to \mathbb{E}^3$ is the trajectory of some point-like object, treated as an observer, then γ could have self-intersections and, correspondingly, Ψ is more easily treated as a lifting of paths.

Definition 4.3. *The unique lifting of paths Ψ or (multiple-valued) section along paths Ψ corresponding to the state vector ψ from conventional quantum mechanics will be called a state lifting (of paths) or a state section (along paths) respectively.*

For brevity, we call, by abuse of the language, a particular value of Ψ, say $\Psi_\gamma(t)$, a *bundle state vector* (at a moment t, or, more precisely, at the (space) point $\gamma(t)$ and at the instant t, i.e. at a spacetime point $(\gamma(t), t)$ if M is treated as a spacetime model).

Since the mappings l_x, $x \in M$, in (4.3) are isomorphisms, the correspondences

$$\text{STATE VECTOR} \Leftrightarrow \text{STATE LIFTING OF PATHS} \Leftrightarrow \text{STATE SECTION ALONG PATHS} \tag{4.4}$$

are bijective (isomorphisms)[31]. Hence, the description of a quantum system via state vectors, or liftings of paths, or sections along paths are equivalent.

On the basis of postulates 4.1 and 4.2, the formalism of conventional quantum mechanics solely concerning the wavefunction (state vector) ψ can be transferred, equivalently, in the Hilbert bundle description in terms of the state lifting Ψ of paths. Equation (4.3) plays a major role in this reformulation. In this direction, our first goal is the Hilbert bundle description of the quantum evolution, i.e. the change of the state liftings/sections Ψ in time. In section 5

[31] In (4.4) the state sections are, generally, multiple-valued sections along paths.

we shall see that the bundle evolution of a quantum system is represented by a suitable linear transport along paths in the system's Hilbert bundle. In the next, second, part of the present investigation the corresponding (bundle) equations of motion will be derived.

We completely understand that even at this early, introductory, stage of our work, many concrete problems arise. They are connected with the transferring and/or interpretation of particular results of conventional quantum mechanics in the case of its bundle (re-) formulation. These questions are, as a rule, out of the scope of the present work, devoted to the general formalism, and have to be considered separately of it. Regardless of this, we want to pay attention to one such problem which may lead to methodological difficulties.

It is well known [1–3] that, in most situations, the wavefunction (vector) of a particle is not localized at a single space point, but it is spread over some space region that could be even the entire space, as in the case of a momentum eigenstate. *Prima facie*, a superficial conclusion can be made that such a state is included in the 'local' Hilbert space at some point, i.e. in the fibre over it. Such a conclusion is generally entirely wrong (unless we are dealing with a state localized at a single point or the base M consists of a single point)! Suppose $\psi \in \mathcal{F}$ is the wavefunction of some quantum system with respect to some observer and $\psi(x, t)$ is its value at a space point x at time $t \in J$. Take the particular choice $M = \mathbb{R}^3$ and let $\gamma : J \to \mathbb{R}^3$ be the observer's trajectory[32]. Since the mappings l_x, $x \in M$ are (isometric) isomorphisms, from (4.3) it follows that the bundle state vector $\Psi_\gamma(t)$ is nonzero if the state vector $\psi(x, t)$ is nonzero. Let $W_t = \mathrm{Supp}\psi(\cdot, t) \subset \mathbb{R}^3 = M$ be the support of $\psi(\cdot, t)$, i.e. $\psi(x, t) \neq 0$ for $x \in W_t$ and $\psi(x, t) = 0$ for $x \notin W_t$ (if $W_t \neq M$). The above implies $\Psi_\gamma(t) \neq 0$ iff $\gamma(t) \in W_t$, in other words $\Psi_\gamma(t) \neq 0$ iff $\pi\big(\Psi_\gamma(t)\big) \in W_t$. Consequently, the nonzero bundle state vectors are spread over the same region (of space) as the 'original' nonzero state vectors. Besides, the nonzero bundle state vectors are in the 'local' Hilbert spaces attached to the corresponding points in W_t; namely, $\Psi_\gamma(t) \neq 0$ is in the fibre $\pi^{-1}(\gamma(t)) \subset \pi^{-1}(W_t)$. In conclusion, the state liftings of paths are localized, i.e. are nonzero, in the same space region as the conventional wavefunctions. Analogous result can be obtained if we take for M other space (-time) models, such as V_4, M^4 etc.

Remark 4.2. The above interpretation of the case $M = \mathbb{R}^3$ (or $M = V_4$ etc) is quite more natural than that of conventional quantum mechanics: the nonzero values of the state liftings/sections are situated in the fibres just above the points at which the wavefunction is nonzero, while its values belong to an abstract Hilbert space which is a highly nonlocal object, associated with the whole space (-time) rather than with some particular point in it. Mathematically our theory is valid if M is arbitrary manifold, but if M is not a space (-time) model, the above (and other) 'nice' interpretation(s) could be lost. For example, if M consists of a single point, $M = \{x\}$, we have $F = F_x = l_x^{-1}(\mathcal{F})$ and, according to note 4.4, we obtain an isomorphic copy in F of the standard quantum mechanics. Now, generally, for $\gamma : J \to \{x\}$ it is hard to find a 'good' interpretation, but if, for example, x is in $\mathbb{R}^3$ (or in V_4 etc), then γ can be interpreted as a trajectory (world line) of an observer situated at a space point x during the whole period of 'observation'.

On one hand, as mentioned earlier, the postulates 4.1 and 4.2 are sufficient for the bundle reformulation of the state vector (wavefunction) formalism. In particular, the probabilistic interpretation of quantum mechanics is retained: since

$$\langle\psi(t)|\psi(t)\rangle = \langle\Psi_\gamma(t)|\Psi_\gamma(t)\rangle_{\gamma(t)} \tag{4.5}$$

which is a corollary of (3.1) and (4.3), the bundle state vector $\Psi_\gamma(t)$ can be interpreted as a probability amplitude. On the other hand, these postulates do not allow us to transfer in

[32] Other choices, such as $M = V_4, M^4, U_4, \ldots$, do not change anything in the next conclusions. The same concerns the interpretation of γ.

the bundle description the predictions of quantum mechanics concerning the observables. To this end, new initial assertions are required. They will be presented in the second part of this investigation in which the exploration of the observables in the bundle approach begins. In short, their essence is the following: in the bundle approach the observables are described via (Hermitian) liftings of paths or (multiple-valued) morphisms along paths (in the bundle of restricted morphisms of the system's Hilbert bundle or in the Hilbert bundle of states respectively) and their mean values (mathematical expectations) are such that they coincide with the mean values of the corresponding (Hermitian) operators representing the same observables in the Hilbert space quantum mechanics. On the grounds of these assertions, the whole machinery of quantum mechanics (of pure states) can be reformulated in terms of fibre bundles. This will be done in the next parts of our work. Due to the just-mentioned coincidence of the mean values of the operators and liftings corresponding to observables, *the predictions of Hilbert space and Hilbert bundle quantum mechanics are absolutely identical, i.e. these are different representations of a single theory, quantum mechanics*. For the bundle description of mixed states, additional postulates are required. They will be presented further. As we shall see, in the bundle approach the mixed states are represented via density liftings of paths (or multiple-valued density sections along paths) such that the mean values of the liftings (or sections) corresponding to observables coincide with the mean values of the corresponding Hermitian operators (in the Hilbert space description) computed by means of the ordinary density operator (matrix). Consequently, as in the case of pure states, now we also have a complete coincidence of the predictions of Hilbert space and Hilbert bundle versions of quantum mechanics.

Beginning with the next section, following the above lines, the purpose of this paper is the bundle formulation of the general formalism of quantum mechanics.

4.4. Preliminary recapitulation

The summary and discussion of the bundle version of quantum mechanics will be presented in the concluding part of this paper. Below we give a short abstract of them with the hope that it will help for the better understanding of our investigation. It also serves as a partial motivation for this paper.

The bundle formulation of quantum mechanics is a purely geometrical version of conventional quantum mechanics to which it is completely equivalent; hence these are simply different 'faces' of a single theory, quantum mechanics. The proposed geometric formulation of quantum mechanics is *dynamical* in a sense that all geometrical structures employed for the description of a quantum system depend on and are determined by the dynamical characteristics of the system. The new form of the theory has three free parameters: the bundle's base M, the set $\{l_x | x \in M\}$ point-trivializing isometric isomorphisms and the path $\gamma : J \to M$. The choice of these objects is external to quantum mechanics and is subject to factors such as the physical interpretation of the theory and its connection with other physical theories. As a working hypothesis, we suggest interpreting M as a space (-time) model and γ as a trajectory (world line) of an observer along which the quantum evolution is studied.

In the Hilbert bundle description the system's Hilbert space is replaced with a suitable Hilbert bundle. In it the system state is represented via appropriate state lifting of paths in the case of pure states or density liftings of paths if the state is mixed. In both cases, the quantum evolution in time is characterized by a linear transport along γ of the state lifting or density liftings in the system Hilbert bundle or in the bundle of its point-restricted morphisms over the base respectively. The corresponding equations of motion are derived. The probabilistic interpretation of quantum theory remains valid.

In the new bundle approach, the observables are described via liftings of paths in the bundle of restricted morphisms over the base in the Hilbert bundle of states. They are so defined that their mean values coincide with the mean values of the corresponding Hermitian operators representing observables in conventional quantum theory. Therefore the physical predictions of the Hilbert space and Hilbert bundle versions of quantum mechanics are identical. The bundle equations of motion, governing the time evolution of observables, are derived.

From the bundle's view-point, an observable is an integral of motion iff it is a constant lifting of paths, namely iff it is linearly transported in the bundle of restricted morphisms over the base in the Hilbert bundle of states with respect to the linear transport induced in this bundle by the evolution transport of state liftings.

We also pay attention to the bundle version of the different pictures of motion. The corresponding equations of motion for the state liftings (or density liftings) and observables are considered in the bundle pictures of motion. We point to an interesting result: in terms of local frames, the bundle Heisenberg picture of motion corresponds to the choice of a suitable normal frame, i.e. a frame in which the matrix of the evolution transport of state liftings is the unit matrix. Since the normal frames are the mathematical objects corresponding to the physical concept of an inertial frame, the above means that the (bundle) Heisenberg picture of quantum mechanics is something like a 'quantum mechanics in a (bundle) inertial frame'.

Finally, we consider problems concerning the role of observers, physical interpretation and possible generalizations of bundle quantum mechanics. In these directions the new form of the theory admits many developments, which is due to the aforementioned three free parameters in it. We point that the presented formalism can be transferred in the relativistic region too.

5. The (bundle) evolution transport

The Hilbert bundle description of quantum evolution of a quantum system is the purpose of this section. More precisely, we want to find the time dependence of the state liftings of paths (or sections along paths) of a system provided the time dependence of its (conventional) wavefunction (state vector) is known[33]. We shall prove that this is achieved via a suitable linear transport along paths, called evolution transport, in the system's Hilbert bundle[34].

According to postulate 4.1, assertion (iii), the evolution of a system in the fibre $\mathcal{F}$ of the system's Hilbert bundle (F, π, M) is given via the evolution operator $\mathcal{U}$ (see section 2). This operator has 'transport-like' properties, similar to (3.19)–(3.21). Indeed, using (2.1), we obtain $\psi(t_3) = \mathcal{U}(t_3, t_2)\psi(t_2) = \mathcal{U}(t_3, t_2)[\mathcal{U}(t_2, t_1)\psi(t_1)]$, $\psi(t_3) = \mathcal{U}(t_3, t_1)\psi(t_1)$, and $\psi(t_1) = \mathcal{U}(t_1, t_1)\psi(t_1)$ for all moments t_1, t_2, t_3 and arbitrary state vector ψ. Hence

$$\mathcal{U}(t_3, t_1) = \mathcal{U}(t_3, t_2) \circ \mathcal{U}(t_2, t_1) \tag{5.1}$$

$$\mathcal{U}(t_1, t_1) = \mathsf{id}_{\mathcal{F}}. \tag{5.2}$$

Besides, by definition, $\mathcal{U}(t_2, t_1) : \mathcal{F} \to \mathcal{F}$ is a linear unitary operator; i.e., for $\lambda_i \in \mathbb{C}$ and $\psi_i(t_1) \in \mathcal{F}$, $i = 1, 2$, we have

$$\mathcal{U}(t_2, t_1)\left(\sum_{i=1,2} \lambda_i \psi_i(t_1)\right) = \sum_{i=1,2} \lambda_i \mathcal{U}(t_2, t_1)\psi_i(t_1) \tag{5.3}$$

$$\mathcal{U}^{\dagger}(t_1, t_2) = \mathcal{U}^{-1}(t_2, t_1). \tag{5.4}$$

From (5.1) and (5.2), evidently, follows

$$\mathcal{U}^{-1}(t_2, t_1) = \mathcal{U}(t_1, t_2) \tag{5.5}$$

[33] The corresponding bundle equations of motion will be derived in the second part of this investigation.

[34] A treatment of quantum evolution as a 'parallel' transport is accepted in [75]. Similar understanding, but in a different context, is maintained in [43].

and consequently

$$\mathcal{U}^{\dagger}(t_1, t_2) = \mathcal{U}(t_1, t_2). \tag{5.6}$$

If one takes as a primary object the Hamiltonian $\mathcal{H}$, these facts are direct consequences of (2.10).

Thus the properties of the evolution operator are very similar to those defining a (Hermitian) linear transport along paths in a Hilbert bundle. In fact, below we show that the bundle analogue of the evolution operator is one kind of such transport.

Along any path γ, we define the bundle analogue of the evolution operator $\mathcal{U}(t, s) : \mathcal{F} \to \mathcal{F}$ as a linear mapping $U_\gamma(t, s) : F_{\gamma(s)} \to F_{\gamma(t)},\ s, t \in J$ such that

$$\Psi_\gamma(t) = U_\gamma(t, s)\Psi_\gamma(s) \tag{5.7}$$

for all instants of time $s, t \in J$. Hence U_γ connects the different time values of the bundle state vectors. Analogously to (5.1) and (5.2), now we have[35]

$$U_\gamma(t_3, t_1) = U_\gamma(t_3, t_2) \circ U_\gamma(t_2, t_1) \qquad t_1, t_2, t_3 \in J \tag{5.8}$$

$$U_\gamma(t, t) = \mathsf{id}_{F_{\gamma(t)}} \qquad t \in J. \tag{5.9}$$

Comparing (5.7) with (2.1) and using (4.3), we find

$$U_\gamma(t, s) = l^{-1}_{\gamma(t)} \circ \mathcal{U}(t, s) \circ l_{\gamma(s)} \qquad s, t \in J \tag{5.10}$$

or

$$\mathcal{U}(t, s) = l_{\gamma(t)} \circ U_\gamma(t, s) \circ l^{-1}_{\gamma(s)} \qquad s, t \in J. \tag{5.11}$$

This shows the equivalence of the description of a quantum evolution via $\mathcal{U}$ and U_γ.

A trivial corollary of (5.10) is the *linearity* of U_γ and

$$U^{-1}_\gamma(t, s) = U_\gamma(s, t). \tag{5.12}$$

As $l_x : F_x \to \mathcal{F},\ x \in M$ are linear isomorphisms, from (5.8)–(5.10) it follows that $U : \gamma \mapsto U_\gamma$ with $U_\gamma : (s, t) \mapsto U_\gamma(s, t) =: U^\gamma_{t\to s} : F_{\gamma(t)} \to F_{\gamma(s)}$ is a *linear transport along paths* in[36] (F, π, M). This transport is *Hermitian* (see section 3). In fact, applying (3.8) to $U_\gamma(t, s)$ and using (5.10), we obtain

$$U^{\ddagger}_\gamma(t, s) = l^{-1}_{\gamma(t)} \circ \mathcal{U}^{\dagger}(s, t) \circ l_{\gamma(t)}. \tag{5.13}$$

So, using (5.6), once again (5.10) and (5.5), we find

$$U^{\ddagger}_\gamma(t, s) = U_\gamma(t, s) = U^{-1}_\gamma(s, t). \tag{5.14}$$

Hence $U_\gamma(t, s)$ is simultaneously Hermitian and a unitary operator, as it should be for any Hermitian or unitary transport along paths in a Hilbert bundle (see section 3). Consequently, U is an isometric transport along paths.

Above we defined the transport U by (5.7) from which (5.7)–(5.14) follow. It is a simple exercise to prove that if U is defined via (5.10), the remaining equations of (5.7)–(5.14) are fulfilled. Consequently, (5.7) and (5.10) are equivalent definitions of the transport U along paths.

[35] Equations similar to (5.8) and (5.8) below are stated in [43] in a case of a Hilbert bundle over the one-dimensional time manifold.

[36] In the context of quantum mechanics it is more natural to define $U_\gamma(s, t)$ from $F_{\gamma(t)}$ into $F_{\gamma(s)}$ instead from $F_{\gamma(s)}$ into $F_{\gamma(t)}$, as is the map $U^\gamma_{s\to t} = U_\gamma(t, s) : F_{\gamma(s)} \to F_{\gamma(t)}$. The latter notation is better in the general theory of transports along paths [34, 35]. Consequently, when applying results from [34, 35], we have to remember that they are valid for the maps $U^\gamma_{s\to t}$ (or $U^\gamma : (s, t) \mapsto U^\gamma_{s\to t}$). That is why for the usage of some results concerning general linear transports along paths from [34, 35] for $U_\gamma(s, t)$ or U_γ one has to write them for $U^\gamma_{s\to t}$ (or U^γ) and then use the connection $U^\gamma_{s\to t} = U_\gamma(t, s) = U^{-1}_\gamma(s, t)$ (or $U^\gamma = U^{-1}_\gamma$). Some results for $U^\gamma_{s\to t}$ and $U_\gamma(s, t)$ coincide but this is not always the case. In short, the results for linear transports along paths are transferred to the case considered in this paper by replacing $L^\gamma_{s\to t}$ by $U_\gamma(t, s) = U^{-1}_\gamma(s, t)$.

Definition 5.1. *The isometric linear transport U along paths, defined via equation (5.7) or (5.10), in the system's Hilbert bundle (F, π, M) of states is called the evolution transport (of the system) or the bundle evolution operator.*

In this way, we see that the evolution transport U is a Hermitian (and hence unitary) linear transport along paths in (F, π, M). Consequently, to any unitary evolution operator $\mathcal{U}$ in the Hilbert space $\mathcal{F}$ there corresponds a unique isometric linear transport U along paths, the evolution transport, in the Hilbert bundle (F, π, M) and vice versa.

Let us summarize. In the Hilbert bundle description, the time evolution of a quantum system is represented by means of the evolution transport along paths in the system's Hilbert bundle. It connects the different time values of the state liftings according to (5.7) along the reference path γ. Equation (5.10) is the link between the evolution transport and the evolution operator; it is equivalent to (4.3) provided (5.7) is postulated.

6. Conclusion

In this paper we have prepared the background for a full self-consistent fibre bundle formulation of nonrelativistic quantum mechanics. For this purpose we replaced the conventional Hilbert space of quantum mechanics with a suitable Hilbert bundle. In this scene, as shown here, the ordinary quantum evolution is described by means of certain linear transports along paths.

It is an advantage of the bundle description of quantum mechanics that it does not make use of any particular model of the base M, but on this model depends the interpretation of 'time' t used. For instance, if we take M to be the three-dimensional Euclidean space $\mathbb{E}^3$ of classical (or quantum) mechanics, then it is natural to identify t with the absolute Newtonian (global) time. However, if M is taken to be the Minkowski four-dimensional space M^4, then it is preferable to take for t the proper time of some (local) observer, but the global coordinate time in some frame can also play the role of t. Principally different is the situation when the pseudo-Riemannian space V_4 of general relativity is taken as M: now t *must* be the local time of some observer as a global time does not generically exist.

Generally, the spacetime model M is external to (bundle) quantum mechanics and has to be determined by another theory, such as special or general relativity. This points to a possible field of research: a connection between the quantities of the total bundle space and a concrete model of M may result in a completely new theory. Elsewhere we shall show that exactly this is the case with relativistic quantum mechanics.

There exist *nonlinear* versions of the standard Schrödinger equation, e.g. those introduced and investigated by Doebner and Goldin (see, for instance, [76–79] and references therein). Naturally one can ask whether the above-proposed description of the quantum evolution as a linear transport along paths is valid *mutatis mutandis* for such equations. The answer is, in the general case, negative, since such a description is due to the existence of a *linear* evolution operator, which does not exist for *nonlinear* equations. At present the bundle description of a quantum evolution governed by a nonlinear variants of the Schrödinger equation is open to investigation. It is likely that for such equations an evolution transport should be defined as a suitable, generally nonlinear, transport along paths in, possibly nonvector, fibre bundles.

The bundle approach to quantum mechanics will be developed in the continuation of this paper. In particular, we intend to investigate the following topics from the novel fibre bundle viewpoint: equations of motion, description of observables, pictures and integrals of motion, mixed states, interpretation of the theory and possible ways for its further development and generalizations.

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